Data Science 1
Probability
Bayes' Theorem

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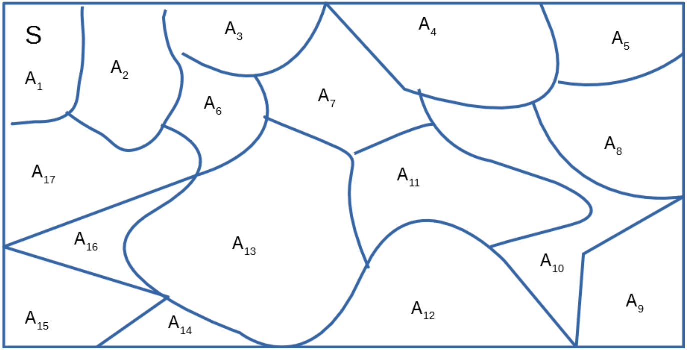
Introduction

Recall the following:

- E is an event in a sample space $\mathcal{S}$.
- $P(\mathcal{S}) = 1$.
- $E \cup F$ is the union of events E and F.
- $E \cap F$ is the intersection of events E and F.
- Probability behaves like area.
- $P(E|F) = \frac{P(E \cap F)}{P(F)}$
- If E and F are independent, then $P(E \cap F) = P(E)P(F)$.

Partition

Suppose we have a **partition** of the sample space $\mathcal{S}$:



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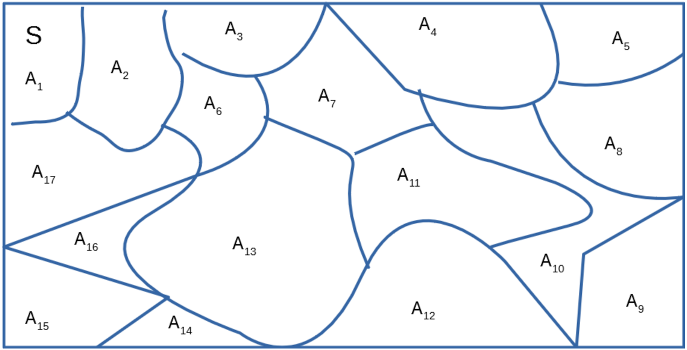
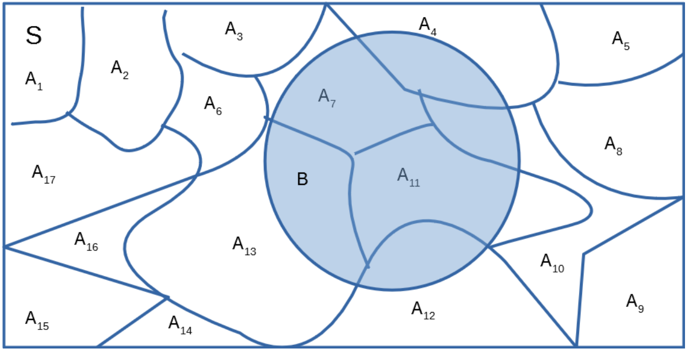


Here $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $P(\mathcal{S}) = \sum_i P(A_i)$.

Total Probability

Suppose $A_1, A_2, \ldots, A_k$ form a partition of $\mathcal{S}$ and B is an event.



Then $P(B) = \sum_i P(B \cap A_i)$.

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This is called the **Law of Total Probability**.

Example

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Write down what we know:

$$P(A) = 0.2,\ P(B) = 0.5,\ P(C) = 0.3$$

$$P(\text{rotten}|A) = 0.03,\ P(\text{rotten}|B) = 0.04,\ P(\text{rotten}|C) = 0.02.$$

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Then,

$$\begin{aligned} P(\text{rotten}) &= P(\text{rotten}|A)P(A) + P(\text{rotten}|B)P(B) + P(\text{rotten}|C)P(C) \\ &= 0.03 \times 0.2 + 0.04 \times 0.5 + 0.02 \times 0.3 \\ &= 0.032 \end{aligned}$$

This is not difficult to work out once you write down what you know.

Bayes' Theorem

Suppose $A_1, A_2, ..., A_k$ form a partition of $\mathcal{S}$ and B is an event. And we know $P(B|A_i)$ and $P(A_i)$ for all i. Then we can use this to "flip" the probability statement.

$$
\begin{aligned}
P(A_j|B) &= \frac{P(A_j \cap B)}{P(B)} \\
&= \frac{P(A_j \cap B)}{\sum_i P(B|A_i)P(A_i)} \\
&= \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}
\end{aligned}
$$

This is known as Bayes' Theorem.

Additional Rule

Recall the following:

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These rules are very helpful.

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We want $P(D|+)$

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$$P(D) = 0.01,\ \ P(+|D) = 0.99,\ \ P(+|D^c) = 0.10 \text{ and } P(D^c) = 0.99$$

Put it all together...

$$\begin{aligned} P(D|+) &= \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)} \\ &= \frac{0.99 \times 0.01}{0.99 \times 0.01 + 0.1 \times 0.99} \\ &= \frac{0.0099}{0.0099 + 0.099} \\ &= 0.05 \end{aligned}$$

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This is an example of where probabilities are not near what you think they would be.

Summary

The law of Total Probability and Bayes' Theorem are incredibly important ideas in Probability and Statistics.

- Total Probability: $P(B) = \sum_i P(A_i \cap B)$
- Bayes' Theorem:

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}.$$

These ideas show up again and again in probability, statistics and data science.