Data Science 1
Statistical Inference
Hypothesis Testing for One Population

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Statistical Inference: Hypothesis Testing for One Population

Introduction

- Hypothesis testing involves drawing inference about two contrasting propositions (hypotheses) relating to the value of a population parameter, one of which is assumed to be true in the absence of contradictory evidence.

- In conducting a hypothesis test, we seek evidence to determine if the assumed hypothesis can be rejected. If not, we can only assume it to be true.

Concepts in Hypothesis Testing

- Null and Alternative Hypotheses
- One-Tailed and Two-Tailed Tests
- Types of Errors
- Significance Level and Rejection Region
- Significance from p-values

Null and Alternative Hypotheses

- The null hypothesis is labelled H_0.
- The alternative hypothesis is labelled H_1 or H_a.
- The null and alternative hypotheses are mutually exclusive; only one of them can be true.
- The null and alternative hypotheses are collectively exhaustive; they are stated to include all possibilities.
- The null hypothesis is assumed to be true.
- The burden of proof falls on the alternative hypothesis.

One-Tailed and Two-Tailed Tests

- The form of the alternative hypothesis can be either one-tailed or two-tailed, depending on what the analyst wants to test.
- One-tailed alternatives are phrased in terms of the $>$ or $<$ signs, whereas two-tailed alternatives are phrased in terms of the $\neq$ sign.

Types of Errors

- Whether or not one decides to reject the null hypothesis or not reject it, it might be the wrong decision.
- One might reject the null hypothesis when it is true or fail to reject it when it is false.
- We commit a Type I error when we incorrectly reject a null hypothesis that is true.
- We commit a Type II error when we incorrectly fail to reject a null hypothesis that is false.

Table: 1

	H_0 is True	H_0 is False
Reject H_0	Type I error	Correct decision
Fail to Reject H_0	Correct decision	Type II error

We assign probabilities to the Type I and Type II errors.

- P(Type I error) $= \alpha$
- P(Type II error) $= \beta$

Significance Level and Rejection Region

- The real question is how strong the evidence in favour of the alternative hypothesis must be to reject the null hypothesis.
- One has to set the probability of a Type I error that one is willing to tolerate.
    - This value is denoted by α and it is most commonly $\alpha = 0.05$.
    - However, $\alpha = 0.01$ and $\alpha = 0.10$ are also frequently used.
    - The value of α is called the significance level of the test.
    - Given the value of α, we use statistical theory to determine critical values(s) and hence the rejection region.
- If the sample falls into the rejection region we reject the null hypothesis, otherwise, we do not.
- Sample evidence that falls into the rejection region is called statistically significant at the α level.

Power of the test

- The probability of the Type II error is denoted by β and $1 - \beta$ is called the power of the test and it represents the probability of correctly rejecting the null hypothesis when it is indeed false.
- If the power of the test is deemed to be too small, it can be increased by taking larger samples.
- Large samples enable us to detect small differences between sample statistics and population parameters with more accuracy.
- However a larger sample size incurs higher costs.

Significance from p-values

- The use of p-values is more more widespread than the use of critical values in hypothesis testing.
- The p-value approach is used to avoid the use of the α level of significance in determining the critical value, and instead simply report *how significant* the sample evidence is.
- The p-value is the smallest level of significance for which the null hypothesis H_0 will be rejected and we use statistical theory to determine it.
- In general, smaller p-values indicate more evidence in support of the alternative hypothesis.
- If a p-value is sufficiently small, almost any decision maker will conclude that rejecting the null hypothesis is the more *reasonable* decision.

Significance from p-values

- How small is a *small* p-value? We usually compare it with some level of significance, namely $\alpha = 0.01$, 0.05 or 0.1. In particular a
    - p-value less than 0.01, provides *convincing* evidence that the alternative hypothesis is true.
    - p-value between 0.01 and 0.05, means there is *strong* evidence in favour of the alternative hypothesis.
    - p-value between 0.05 and 0.10, is a *grey area*.
    - p-value greater than 0.10 is interpreted as *weak* or no evidence in support of the alternative.
- So given that we are testing at particular level of significance, α, the decision rule would be:
    - Reject the null hypothesis if the p-value is less than α.

Design of Hypothesis Tests

There are six steps in the process of hypothesis testing:

1. Formulate the hypotheses.
2. Specify the level of significance, α.
3. Determine which test statistic to use and compute it.
4. Determine the critical value(s) of the test at the α level of significance or determine the *p-value*.
5. State the decision rule.
6. Draw a conclusion.

Hypothesis Tests

1. Population mean
2. Population proportion
3. Population variance

Hypothesis Tests about a Population Mean

- One of the most basic hypothesis tests is a test about a population mean.
- Some situations in which we may perform a test about a population mean are:
    - A manufacturing company wants to determine whether the average thickness of a plastic bottle is 2.4mm.
    - A retail store wants to determine whether the average age of its customers is less than 40 years.
    - An environmental researcher wants to know if the average bacteria count in a city lake exceeds the safety level of 190.

The **population standard deviation σ is known** and the random variable of interest X is normally distributed or the sample size n is *sufficiently large*.

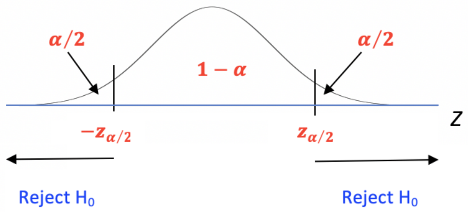
Table: 2

Null Hypothesis	$H_0 : \mu = \mu_0$	$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$	$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
Alternative Hypothesis	$H_1 : \mu \neq \mu_0$	$H_1 : \mu > \mu_0$	$H_1 : \mu < \mu_0$
Level of Significance	α	α	α
Test Statistic	$Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$	$Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$	$Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$
Critical Value Approach Reject H_0 if	$Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$	$Z > z_\alpha$	$Z < -z_\alpha$
P-Value Approach Reject H_0 if	$p - value < \alpha$	$p - value < \alpha$	$p - value < \alpha$

Critical value approach - σ is known

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

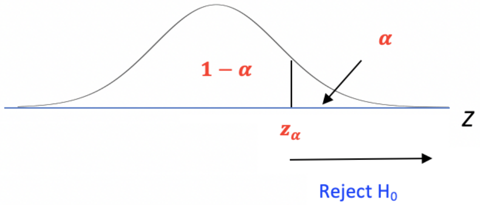


Reject H_0 if $Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$

Critical value approach - σ is known

$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$
$H_1 : \mu > \mu_0$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

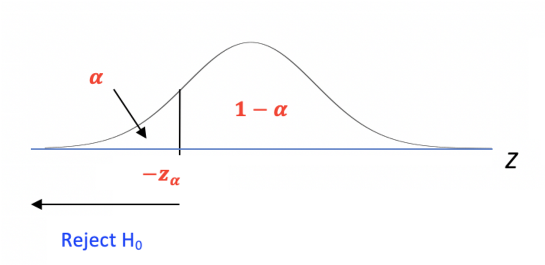


Reject H_0 if $Z > z_\alpha$

Critical value approach - σ is known

$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
$H_1 : \mu < \mu_0$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

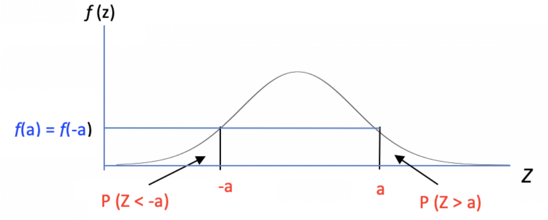


Reject H_0 if $Z < -z_\alpha$

P-value approach - σ is known

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$



a = |numerical value of the test statistic|

$f(z)$ is the density function of the standard normal distribution and $f(-a) = f(a)$

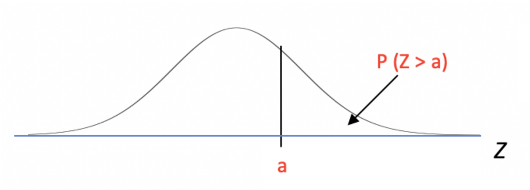
p-value = P(Z > a) + P(Z < -a)

Reject H_0 if $p - value < \alpha$

P-value approach- σ is known

$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$
$H_1 : \mu > \mu_0$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$



a = numerical value of the test statistic

p-value = P(Z > a)

Reject H_0 if $p - value < \alpha$

P-value approach - σ is known

$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
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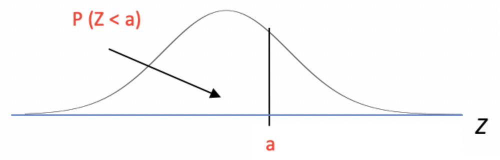
$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$



a $=$ numerical value of the test statistic

p-value $=$ P(Z $<$ a)

Reject H_0 if $p - value < \alpha$

Example 1

- From extensive records, it is known that the duration of treating a particular disease by standard therapy has a mean of 15 days and a standard deviation of 3 days. It is claimed that a new therapy can reduce the treatment time.
- To test this claim the new therapy is tried on a 70 randomly selected patients, their times to recovery are recorded, and the mean recovery time found to be 14.18 days.
- Assume that the standard deviation for this new therapy is also 3 days.
- Is there sufficient evidence to support this claim at the 5% level of significance?

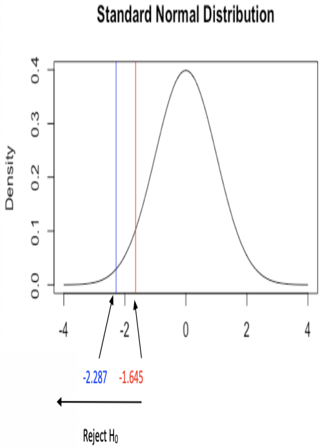
$n = 70, \mu = 15, \sigma = 3, \overline{X} = 14.18, \alpha = 0.05$

Example 1 - R Code

```
> #assumed population mean
> mu <- 15
> #sample size
> n <- 70
> #sample mean
> xbar <- 14.18
> # population standard deviation
> sigma <- 3
>
> #hypotheses
> #H0: mu >= 15
> #H1: mu < 15
>
> #test statistic
> ts <- (xbar - mu)/(sigma/sqrt(n))
> ts
[1] -2.286871
>
> #lower critical value - 5th percentile (0.05th quantile)
> lower_z05 <- qnorm(.05)
> lower_z05
[1] -1.644854
>
> #p-value
> #P(Z < ts)
> pvalue <-pnorm(ts,lower.tail = TRUE)
> pvalue
[1] 0.01110168
```

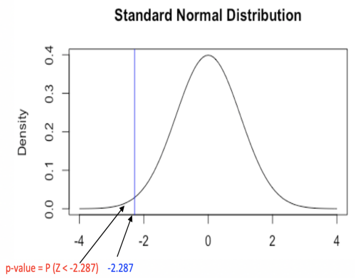
Example 1 - Critical value approach

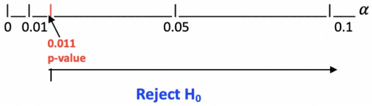
1. Hypotheses:
   $H_0 : \mu \geq 15$
   $H_1 : \mu < 15$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} =$ -2.287
4. Critical value: $-z_{0.05} = -1.645$
5. Decision Rule: Reject H_0 if $Z < -1.645$
6. Conclusion: Since $Z < -1.645$, Reject H_0 at the 5% level of significance. There is sufficient evidence to support the claim that new therapy reduces the treatment time.



Example 1 - P value approach

1. Hypotheses:
   $H_0 : \mu \geq 15$
   $H_1 : \mu < 15$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ = -2.287
4. p-value: $P(Z < -2.287) = 0.011$
5. Decision Rule: Reject H_0 if p-value < 0.05
6. Conclusion: Since p-value< 0.05, Reject H_0 at the 5% level of significance.
   There is sufficient evidence to support the claim that new therapy reduces the treatment time.





The **population standard deviation σ is unknown** and the random variable of interest X is normally distributed.

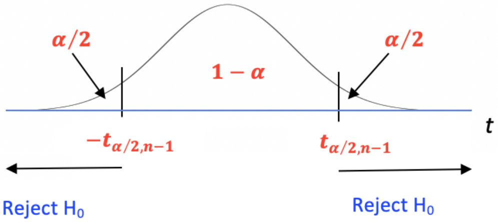
Table: 2

Null Hypothesis	$H_0 : \mu = \mu_0$	$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$	$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
Alternative Hypothesis	$H_1 : \mu \neq \mu_0$	$H_1 : \mu > \mu_0$	$H_1 : \mu < \mu_0$
Level of Significance	α	α	α
Test Statistic	$t = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}}$	$t = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}}$	$t = \frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}}$
Critical Value Approach Reject H_0 if	$t > t_{\alpha/2,n-1}$ or $t < -t_{\alpha/2,n-1}$	$t > t_{\alpha,n-1}$	$t < -t_{\alpha,n-1}$
P-Value Approach Reject H_0 if	$p - value < \alpha$	$p - value < \alpha$	$p - value < \alpha$

Critical value approach - σ is unknown

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

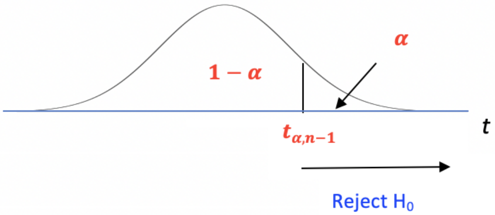


Reject H_0 if $t > t_{\alpha/2,n-1}$ or $t < -t_{\alpha/2,n-1}$

Critical value approach - σ is unknown

$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$
$H_1 : \mu > \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

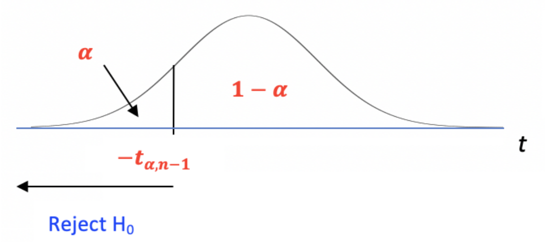


Reject H_0 if $t > t_{\alpha,n-1}$

Critical value approach - σ is unknown

$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
$H_1 : \mu < \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

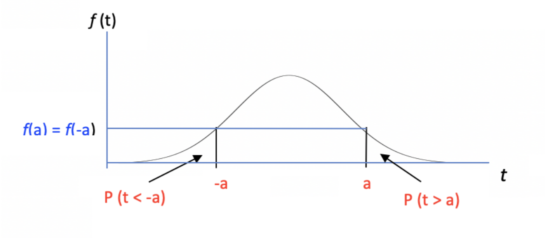


Reject H_0 if $t < -t_{\alpha,n-1}$

P-value approach - σ is unknown

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$



a $=$ |numerical value of the test statistic|

$f(t)$ is the density function of the t-distribution distribution and $f(a) = f(-a)$

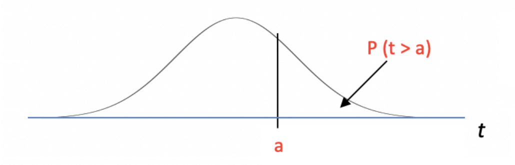
p-value $=$ P(t $>$ a) $+$ P(t $<$ -a)

Reject H_0 if $p - value < \alpha$

P-value approach - σ is unknown

$H_0 : \mu = \mu_0$ or $H_0 : \mu \leq \mu_0$
$H_1 : \mu > \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$



a = numerical value of the test statistic

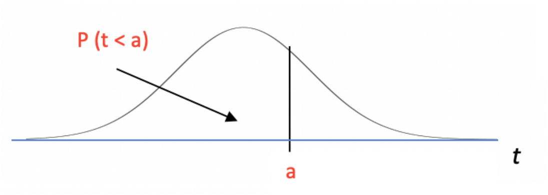
p-value = P(t > a)

Reject H_0 if $p - value < \alpha$

P-value approach - σ is unknown

$H_0 : \mu = \mu_0$ or $H_0 : \mu \geq \mu_0$
$H_1 : \mu < \mu_0$

$$t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$



a $=$ numerical value of the test statistic

p-value $=$ P(t $<$ a)

Reject H_0 if $p - value < \alpha$

Example 2

- A city health department wishes to determine if the mean bacteria count per unit volume of water at a lake beach exceeds the safety level of 190.
- Over a period of week, researchers collected 10 random samples of unit volume and found the bacteria counts to be:
  175, 190, 215, 198, 184, 207, 210, 193, 196, 180
- Assume that the bacteria count per unit volume is normally distributed.
- Is there sufficient evidence at the 5% level of significance to conclude that the safety level has indeed been exceeded?

$\mu = 190$, $n = 10$, $\overline{X}$ and s must be computed from the data, $\alpha = 0.05$

Example 2 - R Code

```
> # read in csv file
> bc <-read.csv("hyp1example2.csv", TRUE)
> #assign the list of bacteria counts to the variable x
> x <- bc$bacteria
>
> #assumed population mean
> mu <- 190
> #sample size
> n <- length(x)
> n
[1] 10
> #sample mean
> xbar <- mean(x)
> xbar
[1] 194.8
> #sample standard deviation
> s =sd(x)
> s
[1] 13.13858
>
> #hypotheses
> #H0: mu <= 190
> #H1: mu > 190
>
> #test statistic
> ts <- (xbar - mu)/(s/sqrt(n))
> ts
[1] 1.155295
>
> #upper critical value - 95th percentile (0.95th quantile)
> upper_t05_9 <- qt(.95,n-1)
> upper_t05_9
[1] 1.833113
```

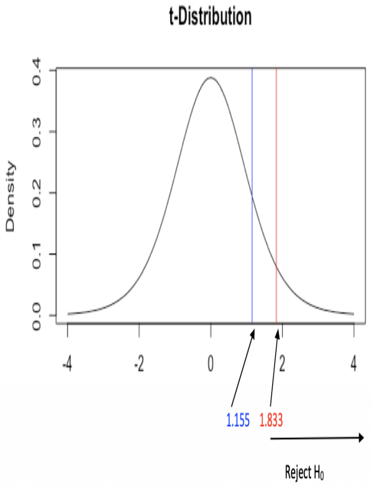
```
> #p-value
> #P(t > ts)
> pvalue <-pt(ts,n-1,lower.tail = FALSE)
> pvalue
[1] 0.1388597
>
> #using t.test in Stats R package
> t.test(x,
+        alternative = "greater",
+        mu = 190,
+        conf.level = 0.95)

        One Sample t-test

data:  x
t = 1.1553, df = 9, p-value = 0.1389
alternative hypothesis: true mean is greater than 190
95 percent confidence interval:
 187.1838      Inf
sample estimates:
mean of x
    194.8
```

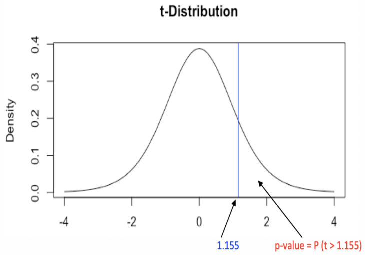
Example 2 - Critical value approach

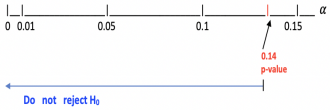
1. Hypotheses:
   $H_0 : \mu \leq 190$
   $H_1 : \mu > 190$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} = 1.155$
4. Critical value: $t_{0.05,9} = 1.833$
5. Decision Rule: Reject H_0 if $t > 1.833$
6. Conclusion: Since $t < 1.833$, do not reject H_0 at the 5% level of significance.
   There is not sufficient evidence to conclude that the safety level has been exceeded.



Example 2 - P value approach

1. Hypotheses:
   $H_0 : \mu \leq 190$
   $H_1 : \mu > 190$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $t = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} = 1.155$
4. p-value: $P(t > 1.155) = 0.1389$
5. Decision Rule: Reject H_0 if p-value < 0.05
6. Conclusion: Since p-value> 0.05, do not reject H_0 at the 5% level or indeed any reasonable level of significance. There is not sufficient evidence to conclude that the safety level has been exceeded.



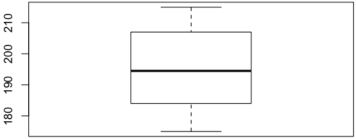


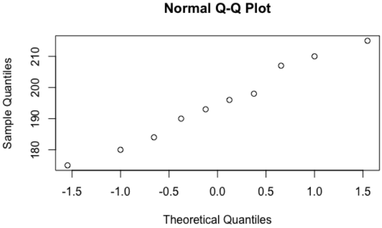
Example 2 - Check for outliers and normality assumption

```
> #check for outliers in data set
> boxplot(x)
> #check for normality of data set
> qqnorm(x)
> shapiro.test(x)

        Shapiro-Wilk normality test

data:  x
W = 0.97152, p-value = 0.9045
```





- No outliers.
- Normality assumption appears to be valid.

Hypothesis Tests about a Population Proportion

Some situations in which we may perform a test about a population proportion are:

- A social science researcher wants to determine if the percentage of families living below the poverty level recorded to be 20% five years ago, has changed.
- A financial researcher wants to determine whether the 60% of companies in the average investment officer's portfolio were profitable last year.
- A quality manager for a large manufacturing company wants to determine whether the proportion of defective items in a batch is less than 0.04.

The sample size *n* is *sufficiently large*.

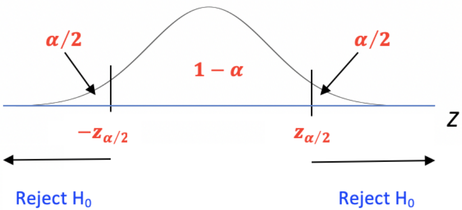
Table: 2

Null Hypothesis	$H_0 : \pi = \pi_0$	$H_0 : \pi = \pi_0$ or $H_0 : \pi \leq \pi_0$	$H_0 : \pi = \pi_0$ or $H_0 : \pi \geq \pi_0$
Alternative Hypothesis	$H_1 : \pi \neq \pi_0$	$H_1 : \pi > \pi_0$	$H_1 : \pi < \pi_0$
Level of Significance	α	α	α
Test Statistic	$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}}$	$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}}$	$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}}$
Critical Value Approach Reject H_0 if	$Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$	$Z > z_\alpha$	$Z < -z_\alpha$
P-Value Approach Reject H_0 if	$p - value < \alpha$	$p - value < \alpha$	$p - value < \alpha$

Critical value approach

$H_0 : \pi = \pi_0$
$H_1 : \pi \neq \pi_0$

$$Z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0,1)$$

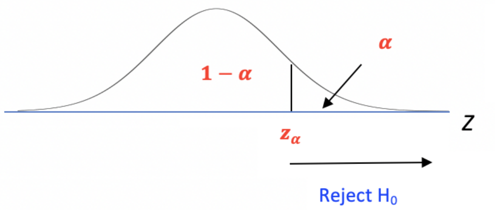


Reject H_0 if $Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$

Critical value approach

$H_0 : \pi = \pi_0$ or $H_0 : \pi \leq \pi_0$
$H_1 : \pi > \pi_0$

$$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0,1)$$

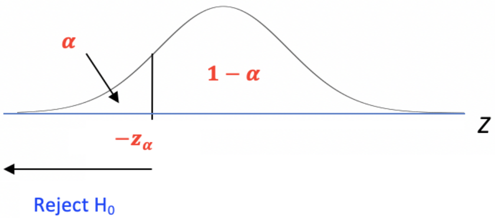


Reject H_0 if $Z > z_\alpha$

Critical value approach

$H_0 : \pi = \pi_0$ or $H_0 : \pi \geq \pi_0$
$H_1 : \pi < \pi_0$

$$Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0,1)$$

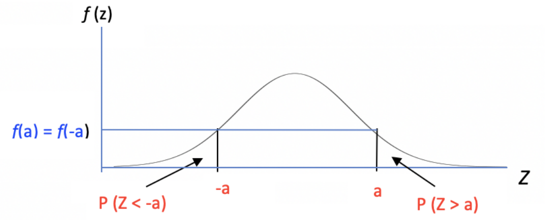


Reject H_0 if $Z < -z_\alpha$

P-value approach

$H_0 : \pi = \pi_0$
$H_1 : \pi \neq \pi_0$

$$Z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0, 1)$$



a $=$ |numerical value of the test statistic|

$f(z)$ is the density function of the standard normal distribution and $f(-a) = f(a)$

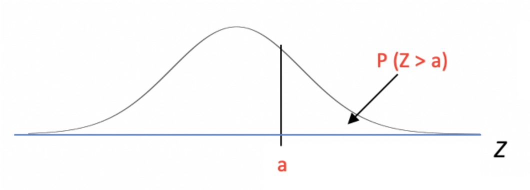
p-value $=$ P(Z $>$ a) $+$ P(Z $<$ -a)

Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \pi = \pi_0$ or $H_0 : \pi \leq \pi_0$
$H_1 : \pi > \pi_0$

$$Z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0,1)$$



a = numerical value of the test statistic

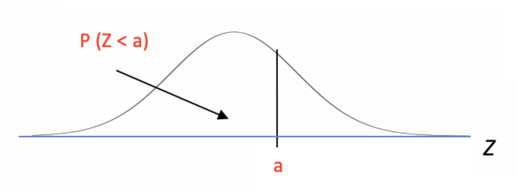
p-value = P(Z > a)

Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \pi = \pi_0$ or $H_0 : \pi \geq \pi_0$
$H_1 : \pi < \pi_0$

$$Z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \sim N(0,1)$$



a = numerical value of the test statistic

p-value = P(Z < a)

Reject H_0 if $p - value < \alpha$

Example 3

- A census conducted 5 years ago in a particular country, recorded that 20% of families in a community located in a large outer western suburb of a major city, lived below the poverty level.
- To determine if this percentage has changed, a random sample of 200 families in this community is studied, and 45 are found to be living below the poverty level.
- Does this finding indicate if there is sufficient evidence at the 5% level of significance that the current percentage differs from the percentage of families who lived below the poverty level 5 years ago.

$\pi = 0.20$, X $= 45$, n $= 200$, $\alpha = 0.05$

Example 3 - R Code

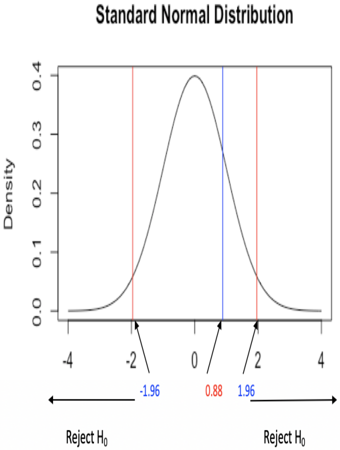
```
> #assumed population proportion
> pi0 <- 0.20
> #sample size
> n <- 200
> # number of families in the sample living below
> # the poverty level
> x <- 45
> #sample proportion
> p <- x/n
> p
[1] 0.225
>
> #hypotheses
> #H0: pi = 0.20
> #H1: pi not= 0.20
>
> #test statistic
> ts <- (p - pi0)/sqrt(pi0*(1-pi0)/n)
> ts
[1] 0.8838835
>
> #lower critical value - 2.5th percentile (0.025th quantile)
> lower_z025 <- qnorm(.025)
> lower_z025
[1] -1.959964
>
> #upper critical value - 97.5th percentile (0.975th quantile)
> upper_z025 <- qnorm(.975)
> upper_z025
[1] 1.959964
```

```
> #determination of the p-value
> #P(Z > abs(ts))
> pv_up <- pnorm(abs(ts),lower.tail=FALSE)
> pv_up
[1] 0.1883796
> #P(Z < -abs(ts))
> pv_lo <-pnorm(-abs(ts),lower.tail=TRUE)
> pv_lo
[1] 0.1883796
> pvalue <-pv_up + pv_lo
> pvalue
[1] 0.3767591
```

Example 3 - Critical value approach

1. Hypotheses:
   $H_0 : \pi = 0.20$
   $H_1 : \pi \neq 0.20$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} = 0.884$
4. Critical values:
   $-z_{0.025} = -1.96, z_{0.025} = 1.96$
5. Decision Rule: Reject H_0 if $Z > 1.96$ or $Z < -1.96$
6. Conclusion: Since $-1.96 < Z < 1.96$, do not reject H_0 at the 5% level of significance.

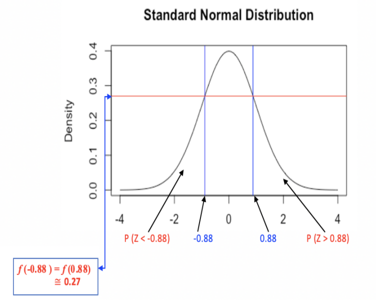
   There is not sufficient evidence to conclude that the percentage of families living below the poverty level has changed.

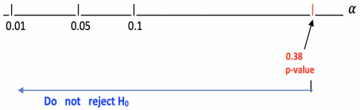


Example 3 - P value approach

1. Hypotheses:
   $H_0 : \pi = 0.20$
   $H_1 : \pi \neq 0.20$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $Z = \frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} = 0.884$
4. p-value:
   $P(Z > 0.88) + P(Z < -0.88) = 0.378$
5. Decision Rule: Reject H_0 if p-value < 0.05
6. Conclusion: Since p-value > 0.05, do not reject H_0 at the 5% level and indeed any reasonable level of significance.

   There is not sufficient evidence to conclude that the percentage of families living below the poverty level has changed.





Hypothesis Tests about a Population Variance

- Hypothesis tests for variance are often applied to quality control.
- For example, we may want to test for variance in thickness of industrial wire or steel tubes to determine if the product is acceptable.
- In using the formula to conduct these tests, it is assumed that the random variable of interest is normally distributed, i.e., the sample is drawn from a normal population.

The random variable of interest X is normally distributed.

Table: 4

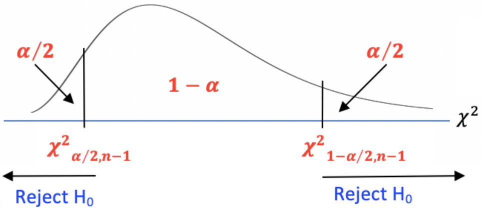
Null Hypothesis	$H_0 : \sigma^2 = \sigma_0^2$	$H_0 : \sigma^2 = \sigma_0^2$ or $H_0 : \sigma^2 \leq \sigma_0^2$	$H_0 : \sigma^2 = \sigma_0^2$ or $H_0 : \sigma^2 \geq \sigma_0^2$
Alternative Hypothesis	$H_1 : \sigma^2 \neq \sigma_0^2$	$H_1 : \sigma^2 > \sigma_0^2$	$H_1 : \sigma^2 < \sigma_0^2$
Level of Significance	α	α	α
Test Statistic	$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$	$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$	$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$
Critical Value Approach Reject H_0 if	$\chi^2 > \chi^2_{1-\alpha/2,n-1}$ or $\chi^2 < \chi^2_{\alpha/2,n-1}$	$\chi^2 > \chi^2_{1-\alpha,n-1}$	$\chi^2 < \chi^2_{\alpha,n-1}$
P-Value Approach Reject H_0 if	$p - value < \alpha$	$p - value < \alpha$	$p - value < \alpha$

Critical value approach

$H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



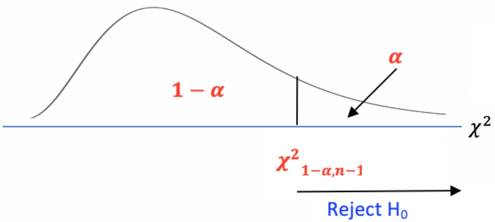
Reject H_0 if $\chi^2 > \chi^2_{1-\alpha/2,n-1}$ or $\chi^2 < \chi^2_{\alpha/2,n-1}$

Critical value approach

$H_0 : \sigma^2 = \sigma_0^2$ or $H_0 : \sigma^2 \leq \sigma_0^2$

$H_1 : \sigma^2 > \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



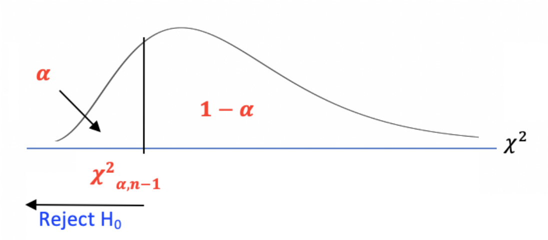
Reject H_0 if $\chi^2 > \chi^2_{1-\alpha,n-1}$

Critical value approach

$H_0 : \sigma^2 = \sigma_0^2$ or $H_0 : \sigma^2 \geq \sigma_0^2$

$H_1 : \sigma^2 < \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



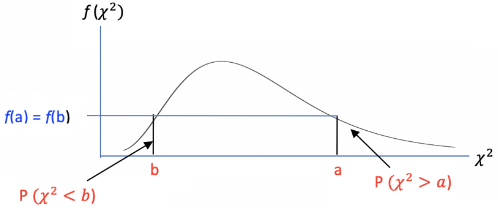
Reject H_0 if $\chi^2 < \chi^2_{\alpha,n-1}$

P-value approach

$H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



a = numerical value of the test statistic

$f(\chi^2)$ is the density function of the chi-square distribution and b is the quantile of the chi-square distribution such that $f(b) = f(a)$. In practice b is determined by trial and error.

p-value $= \mathrm{P}(\chi^2 > a) + \mathrm{P}(\chi^2 < b)$

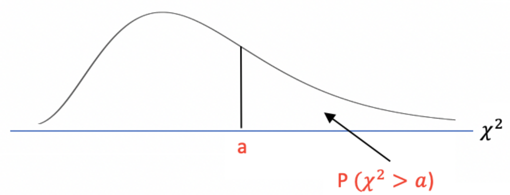
Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \sigma^2 = \sigma_0^2$ or $\sigma^2 \leq \sigma_0^2$

$H_1 : \sigma^2 > \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



a = numerical value of the test statistic

p-value = $P(\chi^2 > a)$

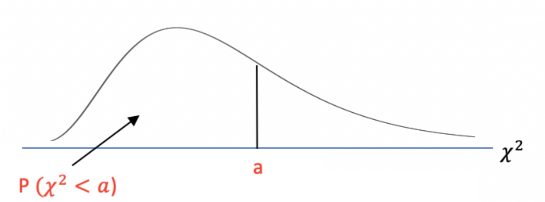
Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \sigma^2 = \sigma_0^2$ or $\sigma^2 \geq \sigma_0^2$

$H_1 : \sigma^2 < \sigma_0^2$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$



a = numerical value of the test statistic

p-value = $P(\chi^2 < a)$

Reject H_0 if $p - value < \alpha$

Example 4

- A business has 100 employees. Because of uncertain demand for its product, the company usually pays overtime in any given week.
- The company assumed that about 50 total hours of overtime per week is required and that the variance on this figure is about 25 squared hours. Company officials want to know whether the variance of overtime hours has changed.
- At hand is a random sample of 16 weeks of overtime data (in hours per week), 57 56 52 44 46 53 44 44 48 51 55 48 63 53 51 50.
- Assuming that hours of overtime are normally distributed, test at the 5% level of significance that the variance of overtime data is 25.

$\sigma^2 = 25$, $n = 16$, s^2 must be computed, $\alpha = 0.05$

Example 4 - R Code

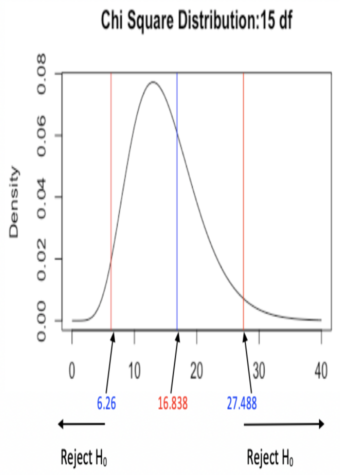
```
> # read in csv file
> ovtm <-read.csv("hyp1example4.csv", TRUE)
> #assign the list of overtime values to the variable x
> x <- ovtm$overtime
>
> #assumed population variance
> sigmasq <- 25
> #sample size
> n = length(x)
> n
[1] 16
> #sample standard deviation
> s <- sd(x)
> s
[1] 5.297405
>
> #hypotheses
> #H0: sigmasq = 25
> #H1: sigmasq not= 25
>
> #chi-square test statistic
> ts <- (n-1)*(s^2)/sigmasq
> ts
[1] 16.8375
>
> #lower critical value - 2.5th percentile (0.025th quantile)
> chi025 <- qchisq(.025,n-1)
> chi025
[1] 6.262138
>
>  #upper critical value - 97.5th percentile (0.975th quantile)
> chi975 <- qchisq(.975,n-1)
> chi975
[1] 27.48839
```

```
> # first determination of the value of the density function
> # for the quantile which is the value of the test statistic
> ts_density <-dchisq(ts,n-1)
> ts_density
[1] 0.06091581
>
> #after trial and error, arrive at a value of the
> #quantile of the chi-square distribution with 15
> #degrees of freedom which gives approximately the
> #same value of the density function as the test statistic
> ts_other = 9.79523
> ts_other_density <- dchisq(ts_other,n-1)
> ts_other_density
[1] 0.06091583
>
> #determination of the p-value
> #P(chisq > ts)
> pv_up <- pchisq(ts, n-1, lower.tail = FALSE)
> pv_up
[1] 0.3286712
> #P(chisq < ts_other)
> pv_lo <- pchisq(ts_other,n-1, lower.tail = TRUE)
> pv_lo
[1] 0.1675811
> #p-value of the test
> pvalue = pv_up + pv_lo
> pvalue
[1] 0.4962522
```

Example 4 - Critical value approach

1. Hypotheses:
   $H_0 : \sigma^2 = 25$
   $H_1 : \sigma^2 \neq 25$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma^2} = 16.838$
4. Critical values:
   $\chi^2_{0.025,15} = 6.262, \chi^2_{0.975,15} = 27.488$
5. Decision Rule: Reject H_0 if $\chi^2 > 27.488$ or $\chi^2 < 6.262$
6. Conclusion: Since $6.262 < \chi^2 < 27.488$, do not reject H_0 at the 5% level of significance.

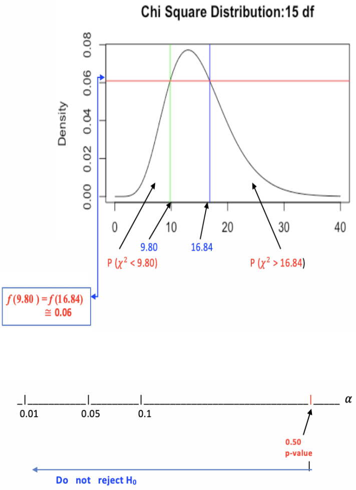
   There is not sufficient evidence to conclude that the variance of overtime hours has changed.



Example 4 - P value approach

1. Hypotheses:
   $H_0 : \sigma^2 = 25$
   $H_1 : \sigma^2 \neq 25$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma^2} = 16.838$
4. p-value:
   $P(\chi^2 > 16.838) + P(\chi^2 < 9.795) = 0.496$
5. Decision Rule: Reject H_0 if p-value < 0.05
6. Conclusion: Since p-value > 0.05, do not reject H_0 at the 5% level and indeed any reasonable level of significance.

   There is not sufficient evidence to conclude that the variance of overtime hours has changed.

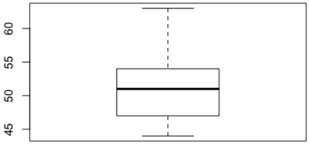


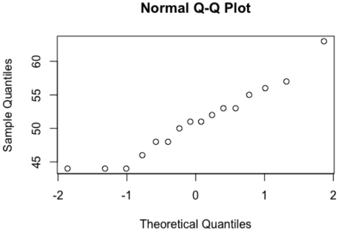
Example 4 - Check for outliers and normality assumption

```
> #check for outliers
> boxplot(x)
> #check for normality of data set
> qqnorm(x)
> shapiro.test(x)

        Shapiro-Wilk normality test

data:  x
W = 0.95004, p-value = 0.4903
```





- No outliers.
- Normality assumption appears to be valid.

Summary

- Concepts in Hypothesis Testing
- Design of Hypothesis Tests
- Hypothesis Tests about the population mean
- Hypothesis Tests about the population proportion
- Hypothesis Tests about the population variance