Data Science 1
Statistical Inference for Two Population Proportions

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Statistical Inference for two Population Proportions

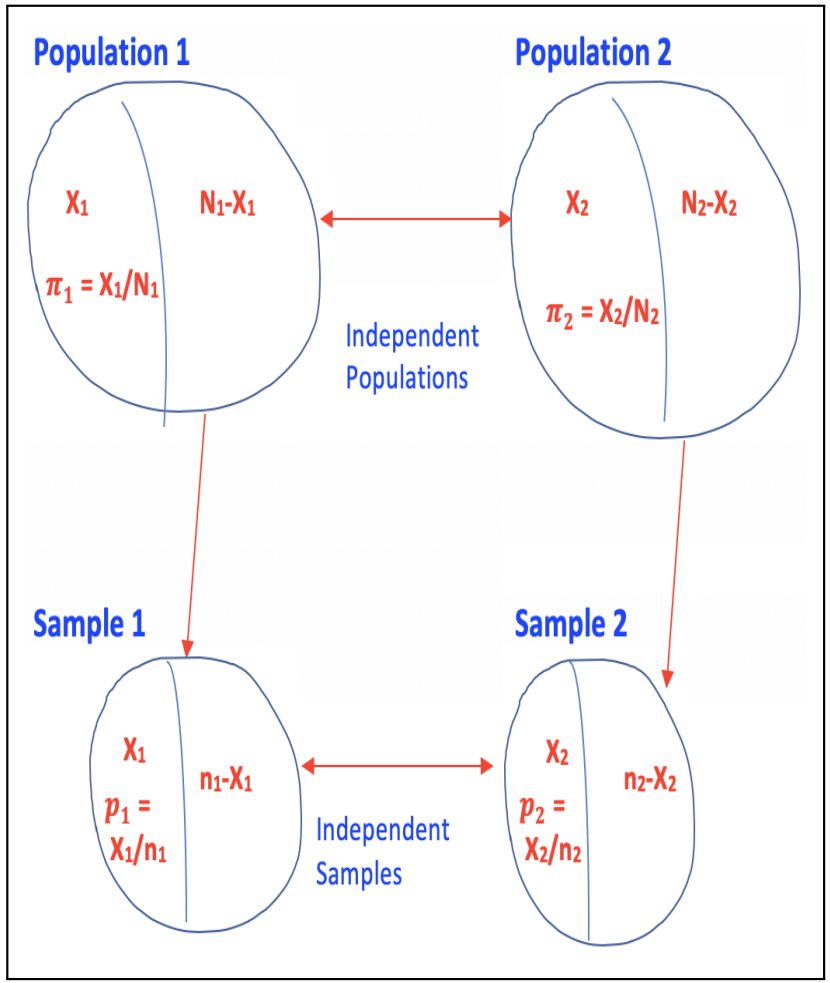
Introduction

There are many situations where we may want to compare proportions between two populations. For example:

- Comparing unemployment rates in rural and urban populations.
- Comparing the proportion of defective items produced by two competing manufacturing processes.

Introduction

- We are interested in a particular characteristic in each of two independent populations.
- Let X_1 and X_2 be the number of cases with this characteristic in the populations which are of size N_1 and N_2, respectively.
- X_1 and X_2 each follows a binomial distribution.
- Let π_1 and π_2 represent the population proportions, where $\pi_1 = X_1/N_1$ and $\pi_2 = X_2/N_2$.
- We take random samples of size n_1 and n_2 from the respective populations and obtain the sample proportions, $p_1 = X_1/n_1$ and $p_2 = X_2/n_2$.
- The point estimator of $\pi_1 - \pi_2$ is $p_1 - p_2$.



Sampling Distribution

- Since the populations are independent:
  - The mean of $p_1 - p_2$ is

$$E(p_1 - p_2) = \pi_1 - \pi_2,$$

    implying that $p_1 - p_2$ is an unbiased estimator of $\pi_1 - \pi_2$.
  - The standard deviation (standard error) of $p_1 - p_2$ is

$$SE(p_1 - p_2) = \sqrt{\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}}.$$

- For large sample sizes, it can be shown that the sampling distribution of $p_1 - p_2$ is approximately normal.
- Then the standardised difference $p_1 - p_2$,

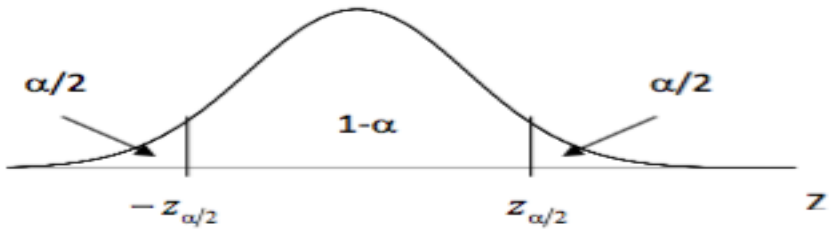
$$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} \sim N(0, 1).$$

Confidence Interval Estimation

Since the standardised difference between the sample proportions, $p_1 - p_2$ follows a $N(0,1)$ distribution, and

$$P\left(-z_{\alpha/2} < Z < z_{\alpha/2}\right) = 1 - \alpha$$

$$\Rightarrow P\left(-z_{\alpha/2} < \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} < z_{\alpha/2}\right) = 1 - \alpha,$$



It can be shown that on making $\pi_1 - \pi_2$ the subject of the formula:
A $100(1-\alpha)\%$ Confidence Interval estimator of $\pi_1 - \pi_2$ is

$$(p_1 - p_2) \pm z_{\alpha/2} \times SE(p_1 - p_2).$$

Replacing π_1 and π_2 with p_1 and p_2, respectively, the standard error term becomes

$$SE(p_1 - p_2) = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}.$$

Table: 1

Null Hypothesis	$H_0 : \pi_1 - \pi_2 = D_0$	$H_0 : \pi_1 - \pi_2 = D_0$ or $H_0 : \pi_1 - \pi_2 \leq D_0$	$H_0 : \pi_1 - \pi_2 = D_0$ or $H_0 : \pi_1 - \pi_2 \geq D_0$
Alternative Hypothesis	$H_1 : \pi_1 - \pi_2 \neq D_0$	$H_1 : \pi_1 - \pi_2 > D_0$	$H_1 : \pi_1 - \pi_2 < D_0$
Level of Significance	α	α	α
Test Statistic	$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)}$	$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)}$	$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)}$
P-Value Approach Reject H_0 if	$p - value < \alpha$	$p - value < \alpha$	$p - value < \alpha$

If $D_0 \neq 0$

$$SE(p_1 - p_2) = \sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}}.$$

If $D_0 = 0$ the common proportion is estimated by the pooled proportion

$$\overline{p} = \frac{n_1}{n_1 + n_2} p_1 + \frac{n_2}{n_1 + n_2} p_2.$$

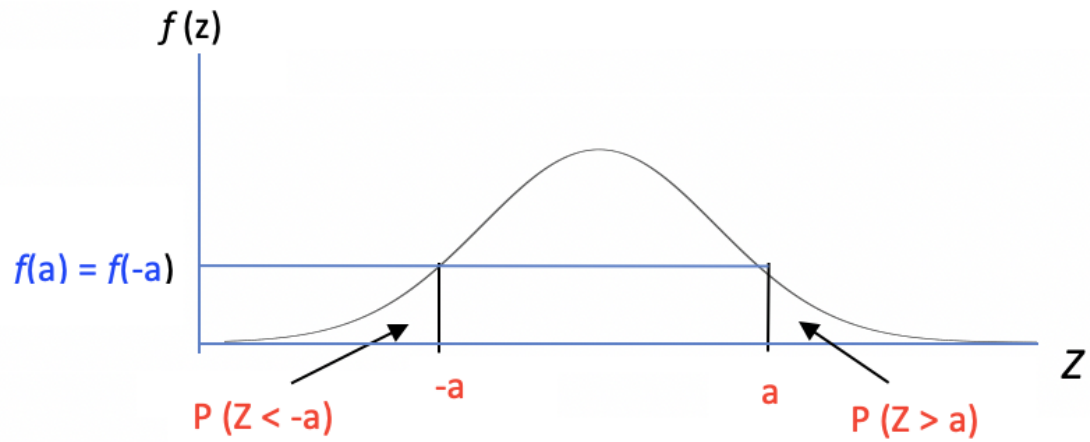
Hence

$$SE(p_1 - p_2) = \sqrt{\overline{p}(1 - \overline{p}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}.$$

P-value approach

$H_0 : \pi_1 - \pi_2 = D_0$
$H_1 : \pi_1 - \pi_2 \neq D_0$

$$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} \sim N(0, 1)$$



a = |numerical value of the test statistic|

$f(z)$ is the density function of the standard normal distribution and $f(-a) = f(a)$

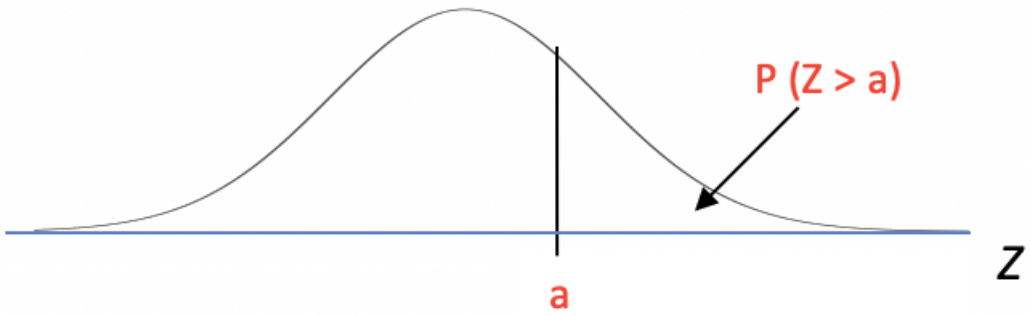
p-value = P(Z > a) + P(Z < -a)

Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \pi_1 - \pi_2 = D_0$ or $H_0 : \pi_1 - \pi_2 \leq D_0$
$H_1 : \pi_1 - \pi_2 > D_0$

$$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} \sim N(0, 1)$$



a = numerical value of the test statistic

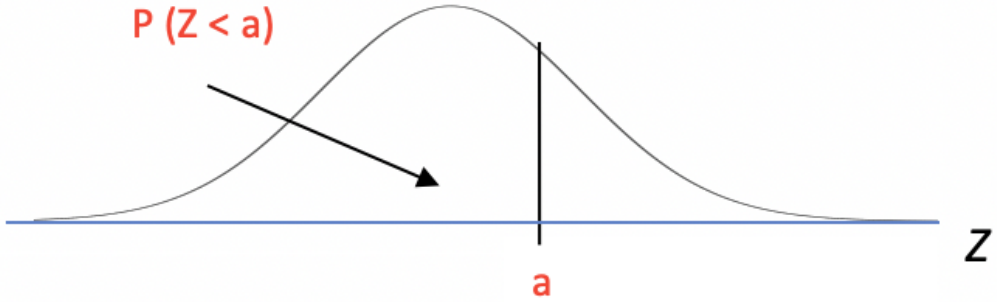
p-value = P(Z > a)

Reject H_0 if $p - value < \alpha$

P-value approach

$H_0 : \pi_1 - \pi_2 = D_0$ or $H_0 : \pi_1 - \pi_2 \geq D_0$
$H_1 : \pi_1 - \pi_2 < D_0$

$$Z = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} \sim N(0, 1)$$



a = numerical value of the test statistic

p-value = P(Z < a)

Reject H_0 if $p - value < \alpha$

A lecturer who prefers innovative teaching methods wishes to compare the effectiveness of teaching Statistics by the traditional classroom lecture method and by the extensive use of audio-visual aids. To do so, 100 students were selected at random from a class of 250 and were assigned to audio-visual instruction. The remaining 150 students are taught Statistics in classroom lectures. At the end of the semester all 250 students are given a test. The number of students from each group who passed the test is given in the following table.

Table: 2

	Audio-Visual Instruction	Classroom Lecture
Pass	63	107
Fail	37	43
Total	100	150

1. Obtain and report a 95% confidence interval estimate of the difference between success rates for the two methods of instruction.
2. Do the data support the hypothesis that a better pass rate is achieved using the classroom lecture method, than is achieved using the audio-visual method?

Example - Part 1

π_1 and π_2: population proportions associated with the classroom lecture and audio-visual instruction methods, respectively.

```
> #Classroom Lecture
> n1 <- 150
> pass1 <- 107
> fail1 <- 43
> #sample proportion
> p1 <- pass1/n1
> p1
[1] 0.7133333
> #Audio-Visual Instruction
> n2 <- 100
> pass2 <- 63
> fail2 <- 37
> #sample proportion
> p2 <- pass2/n2
> p2
[1] 0.63
> #difference bertween sample proportions
> p1-p2
[1] 0.08333333
> #standard error
> se <- sqrt((p1*(1-p1)/n1) +(p1*(1-p1)/n1))
> se
[1] 0.05221608
```

```
> #97.5th percentile of the standard normal distribution
> z025 <- qnorm(.975)
> z025
[1] 1.959964
>
> #Lower confidence limit
> LCL <- (p1-p2) - (z025 *se)
> LCL
[1] -0.01900829
>
> #Upper confidence limit
> UCL <- (p1-p2) + (z025 *se)
> UCL
[1] 0.185675
>
> #95% confidence interval estimate of pi1-pi2
> cbind(LCL, UCL)
              LCL      UCL
[1,] -0.01900829 0.185675
```

- The 95% confidence interval estimate of the difference in population proportions $\pi_1 - \pi_2$ is $[-0.0190, 0.1857]$
- We conclude with 95% confidence that the difference between success rates for the two methods of instruction is between -2% and 19%
- Because the confidence interval contains zero, the null hypothesis $H_0 : \pi_1 - \pi_2 = 0$ is not rejected in favour of the alternative $H_1 : \pi_1 - \pi_2 \neq 0$.

Example: Part 2

π_1 and π_2: population proportions associated with the classroom lecture and audio-visual instruction methods, respectively.

1. Hypotheses:
   $H_0 : \pi_1 - \pi_2 = 0 \quad H_1 : \pi_1 - \pi_2 > 0$
2. Level of significance: $\alpha = 0.05$
3. Test statistic: $Z = 1.384$
4. P-value: $p - value = 0.083$
5. Decision Rule: Reject H_0 if p-value < 0.05
6. Conclusion: Since p-value > 0.05, do not reject H_0 at the 5% level of significance. There is insufficient evidence to conclude that the classroom lecture method is superior to the audio-visual instruction method.

```
> #H0: pi1-pi2 = 0; H1: pi1-pi2 > 0
> #pooled proportion
> pbar <- (n1/(n1+n2)*p1) + (n2/(n1+n2)*p2)
> pbar
[1] 0.68
> sep <- sqrt(pbar*(1-pbar)*(1/n1 + 1/n2))
> sep
[1] 0.06022181
> #Compute test statistic
> ts <- (p1-p2)/sep
> ts
[1] 1.383773
> #p-value = P(Z > ts)
> pvalue <-pnorm(ts,lower.tail = FALSE )
> pvalue
[1] 0.08321395
```

However, if we test the hypothesis at the 10% level of significance, we will reject H_0 and conclude that the classroom lecture method could be superior to the audio-visual instruction method.