**An Inverse Problem Arising from Financial Markets**

Yasushi Ota
Okayama University of Science, Okayama, Japan - yota@soci.ous.ac.jp

**Abstract**

One of the most interesting problems discerned when applying the Black–Scholes model to financial derivatives, is reconciling the deviation between expected and observed values.In our recent work, we derived a new model based on the Black–Scholes model and formulated a new mathematical approach to an inverse problem in financial markets. In this paper, we apply microlocal analysis to prove a uniqueness of the solution and propose numerical method to our inverse problem. First, we explain our model, which is a type of arbitrage model. Next we illustrate our new mathematical approach, and then for space-dependent real drift, we obtain stable linearization and an integral equation. Moreover, by applying microlocal analysis to the integral equation of our inverse problm, we prove our uniqueness of the solution to our new mathematical model in financial markets. Finally, by applying the Tikhonov Regularization mthod and the EM algorithm to our integral equation, we confirm that using market prices of options with different strike prices enables us to identify the term structure of real drift.

**Keywords**: Inverse Problem; Identify Real Drift; The EM Algorithm; The Tikhonov Regularization Method.

## 1. Introduction

In this talk we consider our new model based on the Black-Scholes Model and formulated a new mathematical approach for an inverse problem in financial markets. Financial derivatives are contracts wherein payment is derived from an underlying asset such as a stock, bond, commodity, interest, or exchange rate. An underlying asset $S_t$ at time $t$ is modeled by the following stochastic differential equation:

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \tag{1}$$

where the process $W_t$ is the Brownian motion. The parameters $\mu(t, S)$ and $\sigma(t, S)$ are called the real drift and the local volatility of the underlying asset, respectively.
Black and Sholes first found how to construct a dynamic portfolio $\Pi_t$ of the derivative security and the underlying asset. Their approach is developed in probability theory, and the hedging and pricing theory of the derivative security is established as mathematical finance. By Ito's lemma, the stochastic behavior of the derivative security $u(t, S)$ is governed by the following stochastic differential equation:

$$du = \left(\frac{\partial u}{\partial t} + \mu(t, S)S\frac{\partial u}{\partial S} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 u}{\partial S^2}\right)dt + \sigma(t, S)S\frac{\partial u}{\partial S}dW. \tag{2}$$

In the absence of arbitrage opportunities, the instantaneous return of this portfolio must be equal to the interest rate $r$, the return on a riskless asset, such as a bank deposit. Therefore, this equality takes the form of the following partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma(t, S)^2 S^2\frac{\partial^2 u}{\partial S^2} + (r - \delta)S\frac{\partial u}{\partial S} - ru = 0, \tag{3}$$

where $r$ and the divided rate $\delta$ are the known constants.
Their approach provides us a useful, simple method of pricing with financial derivatives, risk premium, and default probability estimation under the assumption that the risky asset is log-normally distributed. However, theoretical prices of options with different strike prices calculated by the Black-Scholes model differ from real market prices. Specifically, when we apply the Black-Scholes model to default probability estimation, we must be careful of the deviation between the expected and observed values. Merton has formulated a default probability estimation using a model based on Black F and Sholes M (1973) by considering the value of the firm instead of the stock, the firm's debt instead of strike price, and its equity instead of option price.

However, as shown in deriving the Black-Scholes model (see Black F and Sholes M (1973)), under the no arbitrage property of the financial market, the real drift $\mu$ does not enter equation (3). In Mitsuhiro M, Ota Y and Yadohisa H (2012), taking this into account, we have derived the following new model, by using $A_t$ instead of $S_t$:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma(t,A)^2A^2\frac{\partial^2 u}{\partial A^2}+\mu(t,A)A\frac{\partial u}{\partial A}-ru=0. \tag{4}$$

Moreover, in Mitsuhiro M, Ota Y and Yadohisa H (2012) we have established an inverse problem to the reconstruction of the real drift from the observable data, but only an binary option case.
In this talk, we prove a uniqueness for the solution of an inverse problem with respect to the real drift by using an application for microlocal analysis. To give a brief description of our problem, we see the method in Bouchouev I , Isakov V and Valdivia N (2002). In Bouchouev I , Isakov V and Valdivia N (2002), they used the standard linearization method to an option pricing inverse problem and derived a partial differential equation for a principal part $V$. Since, after a change of variables, this equation is reduced to the heat equation with the right-hand side linear with respect to $f$, they wrote the well known integral representation for the solution $V$ as follows:

$$V(\tau,x)=\int_{\mathbf{R}}\int_0^{\tau}\frac{1}{\sqrt{2\pi(\tau-\theta)\sigma_0^2}}e^{-\frac{|x-y|^2}{2\sigma_0^2(\tau^*-\theta)}}w(\theta,y)f(y)d\theta dy, \tag{5}$$

where $w(\tau,y)$ is represented by

$$w(\tau,y)=\frac{s^*}{\sqrt{2\pi\tau\sigma_0^2}}e^{-\frac{|y|^2}{2\sigma_0^2\tau}}. \tag{6}$$

For the above equation they applied the Laplace transform to exactly evaluate an integral with respect to the time. As a result, they derived the integral equation for $f$ of the following form

$$V(\tau,x)=\int_{\mathbf{R}}B(x,y;\tau)f(y)dy \tag{7}$$

with the kernel

$$B(x,y;\tau)=\frac{s^*}{\sigma_0^2\sqrt{\pi}}\int_{\frac{|x-y|+|y|}{\sigma_0\sqrt{2\tau}}}^{\infty}e^{-\theta^2}d\theta \tag{8}$$

given by the error function and proved the uniqueness for the linearized inverse problem. In our case, since the linearized solution $\tilde{V}$ is the following form:

$$\tilde{V}(\tau,x)=\int_{\mathbf{R}}\int_0^{\tau}\frac{1}{\sqrt{4\pi(\tau-s)\sigma_0^2}}e^{-\frac{|y-x|^2}{4(\tau-\theta)\sigma_0^2}-\frac{\mu_0+\sigma_0^2}{2\sigma_0}y}\tilde{w}(\theta,y)f(y)d\theta dy, \tag{9}$$

where $\tilde{w}(\theta,y)$ is the following form

$$\tilde{w}(\tau,y)=\int_0^{\infty}\frac{1}{\sqrt{4\pi\tau\sigma_0^2}}e^{-\frac{|x-y|^2}{4\tau\sigma_0^2}-\frac{\mu_0+\sigma_0^2}{2\sigma_0}y}dx. \tag{10}$$

Therefore we can't derive an integral equation by Laplace transform as (7) that is, in our case $\tilde{w}(\tau,y)$ is not Gauss function as $w(\tau,y)$ but Error function. In the present paper, taking this into account, we shall prove a uniqueness in the inverse problem of the real trend by applying FBI transform to (9).

## 2. Inverse problem of the real drift

As we have seen in Section1, we have derived the new arbitrage model and formulated an inverse option pricing problem for a reconstruction of a real trend in the binary option case. In this section we explain how we can formulate an inverse problem of our new arbitrage model and reconstruct the real drift.
Here, we consider the following problem that the local volatility function $\sigma(t,A)$ is a positive constant $\sigma_0>0$ and the real drift function $\mu(t,A)$ is a space-dependentin our new equation (4) with a suitable condition,

$$u(t,A)|_{t=T}=\max\{A-D,0\}$$

where $D$ is a price of the firm's debt at the maturity date $T$.
By the following changes of variables and substitutions

$$y = \log \frac{A}{D}, \qquad \tau = T - t,$$

$$\mu(y) = \mu(De^y), \qquad U(\tau, y) = u(T-\tau, De^y)/D,$$

one can transform the equation (4) and the initial data into the following inverse problem of the real drift :

$$\begin{cases} \dfrac{\partial U}{\partial \tau} = \dfrac{1}{2}\sigma_0^2 \dfrac{\partial^2 U}{\partial y^2} - \Big(\dfrac{1}{2}\sigma_0^2 - \mu(y)\Big)\dfrac{\partial U}{\partial y} - rU & (y,\tau) \in \mathbf{R} \times (0, \tau^*), \\ U(\tau, y)|_{\tau=0} = \max\{e^y - 1, 0\} & y \in \mathbf{R}, \end{cases} \tag{11}$$

$$U(\tau^*, y) = U^*(y) \qquad y \in \omega \subseteq \mathbf{R}, \tag{12}$$

where $\tau^* = T - t^* > 0$, $t^*$ is the current time and $\omega$ is an interval of $\mathbf{R}$.
Here, we define that the inverse problem of the real drift (11) and (12) seeks $\mu(y)$ from given $U^*(y)$. However, since this inverse problem is a nonlinear, there are difficulties with uniqueness and existence of the solution to one. Therefore, we will formulate the inverse problem of the real drift by means of the method of a linearization in Bouchouev I and Isakov V (1999) and Bouchouev I , Isakov V and Valdivia N (2002).
To linearize around the constant coefficient $\mu_0$, we assume that

$$\mu(y) = \mu_0 + f(y),$$

where $f(y)$ denotes a small perturbation. Thus, we observe

$$U = U_0 + V + \nu,$$

where $U_0$ solves the Cauchy problem (11) with $\mu(y) \equiv \mu_0$, $\nu$ is quadratically small with respect to $f$, and $V$ is the principal part of the perturbed solution $U$. Substituting this into the expression for $u$ and neglecting terms of high order with respect to $f$, we reach the linearized inverse problem of the real drift.

**Linearized Inverse Problem of the real Drift (LIPD)**
*The parameters $\tau^*$, $\mu_0$, $\sigma_0$, and $r$ are given. From the option price $V^*(y) = \{U^*(y) - U_0(\tau^*, y)\}$, identify the perturbation $f(y)$ satisfying*

$$\begin{cases} \dfrac{\partial V}{\partial \tau} - \dfrac{1}{2}\sigma_0^2 \dfrac{\partial^2 V}{\partial y^2} + \left(\dfrac{1}{2}\sigma_0^2 - \mu_0\right)\dfrac{\partial V}{\partial y} + rV = \dfrac{\partial u_0}{\partial y} f(y), \\ V(\tau, y)|_{\tau=0} = 0, \end{cases} \tag{13}$$

$$V(\tau^*, y) = V^*(y). \tag{14}$$

### 3. Main results

In this section we shall prove the uniqueness of the solution to LIPD by using the method of microlocal analysis. Before describing the main theorem, we shall transform the equation of (2.4) into simple form and derive an integral equation of Fredholm type.
We set

$$a_0 = \frac{\sigma_0^2 - 2\mu_0}{2\sigma_0^2}, \quad b_0 = r + \frac{1}{2}\sigma_0^2 a_0^2$$

$$H_a = -\left(\frac{\partial}{\partial y} - a\right)^2 \qquad (a = a_0 - 1)$$

then (13) can be rewritten as

$$\left\{\begin{array}{ll} \left(\dfrac{\partial}{\partial\tau}+\dfrac{1}{2}\sigma_0^2 H_a\right)v(\tau,y)=f(y)\left(e^{-y+b_0\tau}\dfrac{\partial U_0}{\partial y}\right) & (y,\tau)\in\mathbf{R}\times(0,\tau^*), \\ v(\tau,y)|_{\tau=0}=0 & y\in\mathbf{R}, \end{array}\right. \tag{15}$$

where, $v(\tau,y)=e^{-y+b_0\tau}V(\tau,y)$. From the well-known representation of the solution to the Cauchy problem (15), we have the following an integral equation of Fredholm type:

$$v(\tau^*,x)=\int_0^{\tau^*}U_a(\tau^*-s)[w(s,\cdot)f(\cdot)](y)ds. \tag{16}$$

Here

$$(U_a(\tau)\varphi)(y)=\int_R K_a(\tau,y-x)\varphi(x)dx,$$

where

$$K_a(\tau,y)=\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{|y|^2}{4\tau}+ay}$$

and $w(\tau,x)$ is represented the following form:

$$\begin{aligned} w(\tau,x) &:= (U_a(\tau)H_+)(x) \\ &= \int_0^\infty \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{|x-y|^2}{4\tau}+a(x-y)}dx \\ &= \frac{1}{\sqrt{\pi}}e^{\tau a^2}\int_{-\infty}^{\frac{x-2\tau a}{\sqrt{4\tau}}}e^{-\theta^2}d\theta, \end{aligned}$$

where $H_+(x)=1_{[0,\infty]}(x)$.
We will describe results about LIPD in the following Main result.

**Main result**
*Let $\tau^*>0$ and $f(y)\in L^2(\boldsymbol{R})$. Assume that $supp f\subset[-L,\infty)$ with some $L\geq 0$. Then a solution f(y) to the integral equation (16) and hence, to the inverse problem of the real drift (13) and (14) is unique.*

## 4. Numerical example

In this section, we numerically propose our algorithm for reconstruction of a real drift from several discrete option prices. Here, our algorithm is a method to reconstruct a trend coefficient from observed data using the following integral equation derived in Main result.

$$\begin{aligned} &W(\tau^*,x) \\ &= \int_\infty^\infty\int_0^{\tau^*}\frac{1}{\sqrt{4\pi(\tau^*-s)\sigma_0^2}}\exp\left(-\frac{(x-y)^2}{4(\tau^*-s)\sigma_0^2}\right)w(s,y)f(y)dsdy \end{aligned} \tag{17}$$

A discrete representation of the interral equation (17) is given as

$$W(\tau^*,x_j)\approx\sum_{i=1}^N K(x_j,y_i;s)f(y_i)\Delta_y \tag{18}$$

where $y_i(i=1,\cdots,N)$ are points of some interval $\boldsymbol{I}=(-\alpha,\alpha)$, $\Delta y=1/N$ and the points $x_j$ are the measurements points. Here, $K(x_j,y_i;s)$ are the following form

$$K(x_j,y_i;s)=\sum_{l=1}^M\frac{1}{\sqrt{4\pi(\tau^*-s_l)\sigma_0^2}}\exp\left(-\frac{(x_j-y_i)^2}{4(\tau^*-s_l)\sigma_0^2}\right)w(s_l,y_i) \tag{19}$$

Now, we consider the problem of finding the trend coefficient $\boldsymbol{f}$ satisfying the equation

$$\boldsymbol{W} = \boldsymbol{K}\boldsymbol{f} \tag{20}$$

where

$$\boldsymbol{W} = (W(x_1, \tau^*), W(x_2, \tau^*), \cdots, W(x_m, \tau^*))^t$$
$$\boldsymbol{f} = (f(y_1), f(y_2), \cdots f(y_n))^t$$

and $\boldsymbol{K}$ are $m \times n$ matrices of which $(j, i)$-entry is expressed by (19).
In this talk, we consider the following minimization problem: Find $\boldsymbol{f}$ that minimizes the functional

$$\frac{1}{2}||\boldsymbol{K}\boldsymbol{f} - \boldsymbol{W}||^2_{H^1} + \alpha||f||_{L^2}. \tag{21}$$

Moreover, we apply the EM algorithm method to our problem (20).

**References**


Black, F.,& Sholes M. (1973) The pricing of options and corporate liabilities. Journal of Political Economy, 1973, 81, 637-659.

Bonnes A. (1964) Elements of a theory of stock-option value. J. Political Econ, 1964, 72(2), 163-175.

Bouchouev I and Isakov V. (1999) Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets. Inverse Problems, 1999, 15, R95-R116.

Bouchouev I , Isakov V and Valdivia N. (2002) Recovery of volatility coefficient by linearrization. Quantitative Finance, 2002, Vol2, 257-263.

Martinez A. (2002) An introduction to Semiclassical and Microlocal Analysis. 2002, Universitext Springer.

Merton R. (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance, 29, 1974, 449-470.

Mitsuhiro, M., & Ota, Y., & Yadohisa, H. (2012) New mathematical approach for an inverse problem in financial markets. Proceedings of COMPSTAT2012, 2012, 585-594.