Testing spatio-temporal separability by a first-order log-ratio based test. Application to the analysis of wildfire patterns

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Abstract

Testing whether the intensity function of a spatio-temporal point process is separable should be one of the first steps in the analysis of any observed pattern. Under separability, the risk of observing an event at time t is spatially invariant, i.e. the ratio between the intensity functions of the spatio-temporal point process and its spatial marginal does not depend on the spatial location of events. Therefore this work proposes using a no-effect test, which checks the dependence of the ratio function on the spatial locations, to test the separability assumption. To implement the test we introduce a kernel estimator of the log-ratio function, propose a cross-validation bandwidth selector, and discuss two calibration procedures: an asymptotic calibration and a permutation test. The simulation studies conducted to analyze the performance of the test support the use of the permutation test to calibrate the null distribution. Comparison with nonparametric separability tests currently available reported that the no-effect test provides a better calibration under the null hypothesis, and it is competitive in power with the current tests under the alternative hypothesis. Finally, we have applied the separability test to check whether the spatial distribution of wildfires in Galicia (NW Spain) varied over time.

Keywords: intensity function; kernel smoothing; no-effect test; wildfires

1. Introduction

A spatio-temporal point process is a stochastic process governing the location of a random number of events, $\mathbf{S} = \{(\mathbf{x_1}, \mathbf{t_1}), \ldots, (\mathbf{x_N}, \mathbf{t_N})\}$, irregularly placed in $W \times T \subset \mathbb{R}^2 \times \mathbb{R}^+$. Each event in $\mathbf{S}$ is represented by two spatial coordinates, $\mathbf{x_i} = (\mathbf{x_{i1}}, \mathbf{x_{i2}})$, and one temporal coordinate, $\mathbf{t_i}$. Throughout this paper point processes and patterns are denoted in bold capitals, and events are denoted in bold. Spatio-temporal point processes have been increasingly used to address environmental problems such as wildfires or earthquakes, or to deal with disease risk in epidemiological studies.

The spatio-temporal intensity function, which characterizes the first-order structure of any spatio-temporal point process, can be defined as a natural extension of the first-order intensity function of a spatial point process (Diggle, 2013)

$$\lambda(x,t) = \lim_{|dx \times dt| \to 0} \left\{ \frac{E\left[N(dx,dt)\right]}{|dx \times dt|} \right\} \quad (1)$$

where $N(dx, dt)$ represents the number of events in the volume $dx \times dt$, dx is an infinitesimal disc containing the location x, and dt is an infinitesimal interval around time t.

Modeling the first-order intensity function is one of the main issues in the analysis of any observed pattern. Separable models, which assume that the spatio-temporal intensity can be expressed as the product of its spatial and temporal marginals $\lambda(x,t) = \lambda_1(x)\lambda_2(t)$, have been widely used for this purpose given the difficulty of modeling the joint distribution of spatial locations and times of occurrence. As this hypothesis can be quite restrictive and unrealistic, testing separability should be one of the first steps in the analysis of any observed pattern.

Some works have developed separability tests based on the comparison between nonparametric estimators of the separable, $\hat{\lambda}^S(x,t)$, and nonseparable, $\hat{\lambda}^{NS}(x,t)$ intensity functions. Schoenberg (2004) proposed separability tests based on the standardized maximum and minimum absolute distances, a Cramér-von Mises-type statistic (S3) and a log-likelihood separability test (S4). Díaz-Avalos et al. (2013) proposed using a Kullback-Leibler (KL) measure and a Hellinger (H) distance, which measure the difference between separable, p^S, and nonseparable, p^{NS}, kernel estimates of $f(x,t) = \lambda(x,t)/\int_{W\times T} \lambda(x,t)\, dxdt$.

This work introduces a nonparametric separability test with parametric null hypothesis. For this purpose, we take into account that for any separable point process the ratio between the spatio-temporal intensity function and the first-order intensity of the spatial point process comprising the event locations is spatially invariant. Thus we can estimate this ratio and check whether it depends on the spatial locations through a no-effect test.

The paper is organized as follows. In Section 2 we introduce the log-ratio function and its kernel estimator. In Section 3 we introduce the nonparametric separability test. Section 4 discusses the results of the simulation studies conducted to analyze the performance of our separability test. Section 5 illustrates the practical application to the spatio-temporal pattern of wildfires registered in Galicia (NW Spain) during 2006. The paper ends with some conclusions in Section 6.

2. Kernel estimation of the log-ratio function

Let $\mathbf{S} = \{(\mathbf{x_i}, \mathbf{t_i}), i = 1, \ldots, N\} \subset W \times T \subset \mathbb{R}^2 \times \mathbb{R}^+$ be a realization of an inhomogeneous spatio-temporal Poisson point process, and $\mathbf{X} = \{\mathbf{x_i}, i = 1, \ldots, N\} \subset W \subset \mathbb{R}^2$ the corresponding marginal spatial point process, and let $\lambda_0(x,t) = \lambda(x,t)/m$ and $\lambda_{01}(x) = \lambda_1(x)/m$, where $m = \int_T \int_W \lambda(x,t) dxdt = \int_W \lambda_1(x) dx$ is the expected number of events of both the spatial and spatio-temporal point processes, be their densities of event locations. The ratio

$$r(x,t) = \frac{\lambda(x,t)}{\lambda_1(x)} = \frac{\lambda_0(x,t)}{\lambda_{01}(x)}$$

can be seen as a spatio-temporal relative risk function whose control distribution remains constant over time (Sarojinie Fernando and Hazelton, 2014). The log-ratio function $\rho(x,t) = log(\lambda_0(x,t)/\lambda_{01}(x))$ can be estimated by kernel smoothing

$$\hat{\rho}(x,t) = \log \frac{\hat{\lambda}_{0,h_s,h_t}(x,t) + \delta}{\hat{\lambda}_{01,h_s}(x) + \delta} = \log\left(\hat{\lambda}_{0,h_s,h_t}(x,t) + \delta\right) - \log\left(\hat{\lambda}_{01,h_s}(x) + \delta\right) \tag{2}$$

where $\hat{\lambda}_{0,h_s,h_t}(x,t)$ and $\hat{\lambda}_{01,h_s}(x)$ are the kernel estimators of the spatio-temporal and spatial densities of event locations, respectively. $h_s = (h_{s1}, h_{s2})$ denotes the main diagonal of the common diagonal bandwidth matrix used in the spatial component of the kernel estimators of $\lambda_0(x,t)$ and in the kernel estimator of $\lambda_{01}(x)$, and h_t is the scalar bandwidth for the temporal component in the numerator of $\hat{\rho}(x,t)$. In this work the optimal bandwidth was selected by least-squares cross-validation (LSCV). δ is a stabilizing constant that reduces the negative effect of data sparseness on the log-ratio estimator.

3. The separability test

For a separable spatio-temporal point process, $\lambda(x,t) = \lambda_1(x)\lambda_2(t)$, therefore the log-ratio function, $\rho(x,t) = log\left(\lambda(x,t)/\lambda_1(x)\right)$ does not depend on the spatial locations, x, for any $t \in T$. Thus we have a regression problem where the log-ratio function evaluated at each event, $Y = \{y_i = \rho(\mathbf{x_i}, \mathbf{t_i}), i = 1, \ldots, n\}$ is a response variable that may depend on the spatial covariate $X = \{x_i = (x_{i1}, x_{i2}), i = 1, \ldots, n\}$ comprising the event locations, and we test for the effect of X on Y. Following Bowman and Azzalini (1997), we shall discriminate between two competing models

$$\mathcal{H}_0 : E\left[y_i|x_i\right] = \mu. \quad \mathcal{H}_1 : E\left[y_i|x_i\right] = m(x_i)$$

where, for any $x = (x_1, x_2) \in \mathbb{R}^2$, $m(\cdot)$ is an unknown smooth function, which can be estimated by kernel regression (Nadaraya, 1964; Watson, 1964)

$$\hat{m(x)} = \hat{m}(x_1, x_2) = \frac{\sum_{i=1}^{n} w_{g_1}(x_{i1} - x_1) w_{g_2}(x_{i2} - x_2) y_i}{\sum_{i=1}^{n} w_{g_1}(x_{i1} - x_1) w_{g_2}(x_{i2} - x_2)} \tag{3}$$

where the kernel, $w(\cdot)$, is a univariate symmetric density function and $g = (g_1, g_2)$ is the vector of smoothing parameters. Three alternative procedures have been commonly used to select this parameter: (i) bandwidth selector associated to the approximate degrees of freedom, df, of the regression errors, (ii) least-squares cross-validation, CV, and (iii) an AICC-based method.

Once computed $\hat{y} = \sum_{i=1}^{n} y_i$, which is the empirical estimator of μ in $\mathcal{H}_0$, and the regression function, $\hat{m}(\cdot)$ in $\mathcal{H}_1$, we compute the residual sum of squares for the null, RSS_0, and alternative, RSS_1, models and define the generalized test statistic

$$F = \frac{\left(RSS_0 - RSS_1\right) / \left(df_1 - df_0\right)}{RSS_1 / df_1} \tag{4}$$

where df_0 and df_1 denote the degrees of freedom for the residuals under each hypothesis. This separability test is refereed here as *F-test*.

Bowman and Azzalini (1997) proposed two calibration procedures: (i) a χ^2 approximation of the null distribution of F, which can be used when the errors of the regression model are normal, and (ii) a computationally intensive procedure based on permutation tests otherwise.

4. Simulation studies

We have conducted simulation studies to determine the best calibration procedure and to compare the F-test with the nonparametric tests proposed by Schoenberg (2004) and Díaz-Avalos et al. (2013). We simulated spatio-temporal point patterns with intensity function

$$\lambda_1(x, t) = 1000(1 - \epsilon)\phi_{2,(\mu,\Sigma)}(x) e^{-t} + 450\epsilon \left(\frac{1}{4}\right)^3 I^3_{[0.5,0.75]}(x, t) \tag{5}$$

where $\phi_{2,(\mu,\Sigma)}$ is the bivariate normal density with mean $\mu = (0.5, 0.5)$ and covariance matrix $\Sigma = 0.05 I_2$, where I_2 is the two-dimensional identity matrix. The degree of departure from separability is determined by ϵ, which ranges from 0 to 0.5 at 0.05 intervals. For each ϵ we computed the test statistic, $\hat{F}$, and the corresponding empirical p-value for 1000 realizations of the simulated pattern. The probability of rejecting separability at any significance level, α, was obtained as the proportion of p-values smaller than α. To implement the test we applied the LSCV bandwidth selector for the kernel log-ratio function (2), and the three bandwidth selectors available in the *sm* package (Bowman and Azzalini, 2014) of R (R Core Team, 2014) for the the kernel regression function (3).

Table 1 shows the probabilities of rejecting separability under $\mathcal{H}_0$, $\epsilon = 0$, with known and kernel log-ratio function. The permutation test provided a good calibration of the null distribution with the theoretical log-ratio, and accurate estimates of the distribution of F when we use a kernel estimator of the log-ratio function. The χ^2 approximation provided accurate estimators for the size of the test with theoretical log-ratio

Table 1: Probability of rejecting separability, $\hat{p}$, and 95% confidence interval, $[\hat{p}_{0.025}, \hat{p}_{0.975}]$, under $\mathcal{H}_0$ for $\alpha = 0.05$. Performance of the test with known $\rho(x,t)$ and kernel $\hat{\rho}(x,t)$ log-ratio functions. Comparison of calibration methods, χ^2: χ^2 approximation of the null distribution, P. test: $B = 1000$ realizations of the permutation test. Bandwidth selector for the kernel regression function (3) in columns.

		$\rho(x,t)$			$\hat{\rho}(x,t)$		
Calibration		df	CV	AICC	df	CV	AICC
χ^2	$\hat{p}$	0.040	0.087	0.058	0.103	0.125	0.080
	$\hat{p}_{0.025}$	0.028	0.070	0.044	0.084	0.105	0.063
	$\hat{p}_{0.975}$	0.052	0.104	0.072	0.122	0.145	0.097
P.test	$\hat{p}$	0.063	0.050	0.041	0.060	0.057	0.056
	$\hat{p}_{0.025}$	0.048	0.036	0.029	0.045	0.043	0.042
	$\hat{p}_{0.975}$	0.078	0.064	0.053	0.075	0.071	0.070

Table 2: Comparison of the F-test with previous tests. Probability of rejecting separability ($\hat{p}$, and 95% confidence interval, $[\hat{p}_{0.025}, \hat{p}_{0.975}]$), under $\mathcal{H}_0$ at significance level $\alpha = 0.05$.

	F-test			Previous tests			
	df	CV	AICC	S_3	S_4	KL	H
P	0.06	0.057	0.056	0.122	0.053	0.011	0.009
$\hat{p}_{0.025}$	0.045	0.043	0.042	0.102	0.039	0.005	0.003
$\hat{p}_{0.975}$	0.075	0.071	0.07	0.142	0.067	0.017	0.015

function, but tend to be too conservative for the kernel log-ratio function. We have also seen that the power of the test increases with the departure from separability with both calibrations, and the permutation test was competitive or even more powerful that the χ^2 approximation.

Table 2 shows the results of the comparison with the separability tests currently available. S_4 provided a good calibration of the null distribution, S_3 yielded probabilities of rejecting $\mathcal{H}_0$ larger than the nominal significance levels when $\epsilon = 0$, while KL and S were more conservative than the F-test, which provided a good calibration of the null distribution. We have also seen that the F-test is mor powerful than S_4.

5. Testing separability in the spatio-temporal pattern of wildfires registered in Galicia

Wildfire is the most ubiquitous natural disturbance in the world and represents a problem of considerable social and environmental importance. Particularly in Galicia (NW Spain) arson wildfires are the main cause of forest destruction.

In this section we consider the dataset comprising the spatial locations and time of occurrence for the wildfires registered in Galicia during 2006 classified by cause (arson, natural, negligences, reproductions and fires with unknown cause) (Figure 1). The F-test detected departure from separability in the five wildfire patterns. These results indicate that the spatial distribution of the different types of wildfires varied over time and support the need of nonseparable models to estimate their spatio-temporal intensity.

6. Conclusions

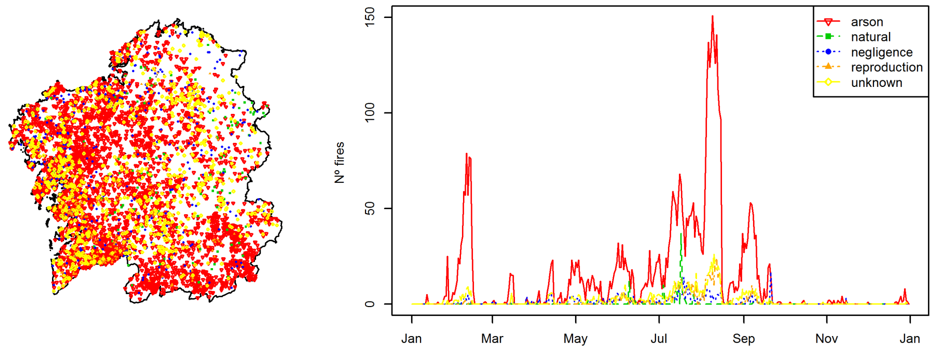

Figure 1: Spatial and temporal patterns of wildfires registered in Galicia during 2006 classified by cause

Testing the separability assumption should be among the first steps in the analysis of spatio-temporal point processes, due to its implication for the modeling of the spatio-temporal intensity. Up to date, this hypothesis has been tested by nonparametric tests calibrated through Monte-Carlo simulations of separable point processes. Taking into account that for a separable point process the risk of observing an event at time t is spatially invariant, we propose a no-effect test that checks whether the log-ratio between the spatio-temporal and spatial intensity functions depend on the spatial locations.

The results of the simulation study conducted to analyze the performance of the test support the use of a permutation test as calibration procedure. Comparison with prior separability tests reported a better behavior of the F-test under both the null and alternative hypotheses. The faster rate of convergence of parametric estimators may explain the good performance of the F-test, with parametric null hypothesis, in comparison with the former, which have nonparametric null hypothesis.

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