A Proposal of Confidence Bands for the Estimation of Expected Discounted Warranty Costs for Coherent Systems under Minimal Repair

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Abstract

This work develops simultaneous confidence bands for the expected discounted warranty costs in coherent systems under physical minimal repair, by means of computer intensive methods based on resampling. For this purpose, it is shown that, under the framework of the Martingale processes and the central limit resampling theorem (CLRT) over stochastic processes proposed by Belyaev (2000) and Belyaev & Seleznjev (2000), that the discounted warranty costs processes satisfy the conditions set by the central limit resampling theorem (CLRT). Additionally, a simulation study is performed on the most relevant variables, such that the finite sample features of the proposed bands may be assessed, based on their actual coverage probabilities. The results in the considered scenarios show that resampling-based simultaneous confidence bands have coverage probabilities that are close to the nominal coverage. In particular, the agreement is good when there are 100 systems or more where a large number of resamples are used for the approximation.

Keywords: central limit resampling theorem; coverage probability; general lifetime model; resampling; weakly approaching.

1. Introduction

When a manufacturer puts a new product on the market, it is expected that a warranty program will come along with the product, which could become a great cost if it is a low-quality product (Thomas 2005). Bai & Pham (2006) highlight that warranty analysis is very important for cost modeling, especially for discounted costs (Blischke & Murthy 1994; Chien 2005; Ja et al. 2002; Murthy & Djamaludina 2002; Nguyen & Murthy 1984). Discounted warranty cost models consider the product age and provide a proper measurement of the costs involved in the warranty program, given that they can be treated as random cash flows. Therefore, it is possible to model their evolution throughout the lifetime of the product under warranty, as well as to estimate the necessary fund reserve levels to meet future warranty claims. Several aspects regarding discounted warranty costs and their corresponding reserves have been studied by Bai & Pham (2004), Duchesne & Marri (2009), Ja et al. (2002), Jain & Maheshwari (2006), Patankar & Mitra (1995), and Thomas (1989).

In practice, many products consist of several components, that is, the product can be seen as a system. If every component has its own warranty, they can be combined to produce one warranty for the system, in which it is necessary to keep in mind both the system structure and warranty service costs at its component level (Thomas, 1989). Several previous works have considered warranty cost analysis for component systems (Ritchken & Fuh, 1986; Chukova & Dimitrov, 1996; Hussain & Murthy, 1998; Bai & Pham, 2006; Balachandran et al., 1981; Jung, et al., 2010). There are many ways to model the impact of repair actions over system failure times. In literature, it is frequently assumed that repairing a system leaves it as good as new (Block et al., 1985). Nevertheless, this hypothesis and its implications have been criticized by many authors on the argument that repairing can only, in many practical cases, restoring the system back to the performance conditions right before the failure (Block et al., 1985; Ascher, 1968; Ouali et al., 2011), which is called minimal repair. In complex systems, repairing is frequently assumed as minimal (Blischke & Murthy, 1994). The definition of the state of the system immediately before failure depends on the information level one has about the system (Aven & Jensen, 1999), so that minimal repairs are classified into two types:

statistical minimal repair, which implies replacing the full system for another one just as old as the other one would be if it had not failed (Nguyen & Murthy, 1984; Aven, 1983; Sheu et al., 1995; Ja et al., 2001), and physical minimal repair, in which the system is supposed to be observed at its component level and, therefore, only the critical component that caused the system to fail, gets minimal repair (Aven & Castro, 2008; Gonzalez & Bueno, 2011).

Gonzalez & Bueno (2011) propose a Martingale estimator for the expected discounted warranty costs for a coherent system under physical minimal repair and include the calculation of specific confidence limits which do not make up a simultaneous confidence band, given that the aforementioned set of limits generally has no correct coverage probability (Fleming & Harrington, 1991). The main purpose of constructing simultaneous confidence bands is to assess an estimator's precision, which can be described by the distribution (or a function of it) of that estimator's deviations from its real value (Belyaev, 2007). The problem is that the aforementioned distribution is unknown, even if asymptotic convergence results of distribution can be obtained (Gonzalez & Bueno, 2011). In the practice, sample sizes are not always large enough for those approximations to work properly. In general, computer intensive (CI) methods provide a way to find asymptotically precise approximations of the estimator deviation distributions from the real unknown parameters (Belyaev, 2000). The bootstrap, introduced by Efron (1979) is a rather universal method, nevertheless, the need to find a proper estimator of the real parameter which can describe data distribution, may be a difficult problem, which is why resampling can be used alternatively (Belyaev, 2000).

Resampling is used in this work to develop simultaneous confidence bands for the mean function of the discounted warranty costs of a system under physical minimal repair. For this, based on the theoretical framework of Martingale process and the Central Limit of Resampling Theorem (CLRT) over stochastic processes proposed by Belyaev (2000) and Belyaev & Seleznjev (2000), proof is presented that the discounted warranty costs processes under the general lifetime model comply with CLRT conditions. In addition, a simulation study was conducted on the most relevant variables to test the finite sample features of the proposed simultaneous confidence bands by means of the actual coverage probability.

Section 2 presents the theoretical framework that is necessary in the development of this work. The proposal of constructing simultaneous confidence bands is developed in Section 3. In Section 4, the performance of the proposed simultaneous confidence bands is assessed by means of a simulation study. Section 5 presents the work's most important conclusions and recommendations.

2. Theoretical Framework

In this work it is assumed that there is a coherent system under physical minimal repair, that is, under the physical approach of the general failure model (Aven & Jensen, 1999). Next section summarizes some theoretical results which are necessary for the development of the remaining sections in the work.

2.1. Physical Minimal Repair Model for a Coherent System and Discounted Warranty Costs

Let us suppose a system with m components, where T is the system lifetime, S_i is the lifetime of component i, $i = 1, \ldots, m$ and $\tilde{N}_t$ is the number of minimal repairs of the system in the interval $[0, t]$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$,

$$\mathcal{F}_t = \sigma \left(\tilde{N}_s, I\left(S_i > s\right), 0 \leq s \leq t, i = 1, \ldots, m \right), \tag{1}$$

where $I(A)$ is the indicator of the event A. The indicator $I(A)$ is equal to 1 when the event A happens and it is 0 otherwise. Therefore, the system repair/failure process is observed at the level of its m components. Suppose the following conditions are hold:

a) $0 < S_i < \infty$ P-a.s., $i = 1, \ldots, m$, where P-a.s. denotes that an event E happens almost surely with respect to the probability P.

b) For every $i \neq j$, $P\left(S_i = S_j\right) = 0$, that is, there are not two components failing simultaneously.

c) All lifetimes S_i are totally inaccessible $\mathcal{F}_t$-stopping times, and consequently all the compensators A^i of the respective simple counting processes $N_t^i = I\left(S_i \leq t\right)$, are continuous P-a.s.

Given that the condition b) implies that the observable failure blocks (sets of components which can fail simultaneously) in an instant t consist of only one of the m components, the m (only P-a.s.) compensators A_t^i corresponding to the simple counting processes $N_t^i = I(S_i \le t)$ are obtained from Doob-Meyer decomposition,

$$N_t^i = A_t^i + M_t^i, \quad M_t^i \in \mathcal{M}_0^2, \quad i = 1, \dots, m, \tag{2}$$

where,

$$A_t^i = \int_0^t I(S_i > s) \lambda^i(s) \, ds < \infty \; P\text{-a.s.}, \quad i = 1, \dots, m, \tag{3}$$

and A_t^i the failure rate function for component i. Likewise, the system failure indicator process $N_t = I(T \le t)$ is given by

$$N_t = A_t + M_t, \quad M_t \in \mathcal{M}_0^2, \tag{4}$$

where, assuming that at $\{T > t\}$, T has a $\mathcal{F}_t$-intensity λ_t, the compensator of $N_t = I(T \le t)$ corresponds to

$$A_t = \int_0^t I_{\{T>s\}} \lambda_s ds < \infty \; P\text{-a.s.} \tag{5}$$

Aven (1983) established that the component i contributes to system failure only after its critical level Y_i, that is, the first instant after which failure of component i causes the system failure. Let K_j, $j = 1, \dots, r$, be the minimal cut sets, that is, each K_j is a minimal subset of components whose joint failure causes the system to fail (Barlow & Proschan, 1981). Then, the critical level Y_i is the $\mathcal{F}_t$-stopping time

$$Y_i = \min_{j:i \in K_j} \max_{u \in K_j - \{i\}} S_u. \tag{6}$$

It assumes that $Y_i = \infty$ if either the system or component i fails before the latter becomes critical ($S_i \le Y_i$ or $T \le Y_i$). Since there are not two components failing simultaneously, the system will fail at time t when the first critical component for the system at t^- fails at t. Therefore, the system lifetime can be written as

$$T = \min_{i:Y_i < \infty} S_i, \tag{7}$$

Gonzalez & Bueno (2011) show that the compensator in (5) has the following form,

$$A_t = \int_0^t I(T > s) \sum_{i=1}^m I(Y_i < s) \lambda^i(s) \, ds. \tag{8}$$

From (5) and (8), it follows clear that the $\mathcal{F}_t$-intensity of the system on $\{T > t\}$ is given by

$$\lambda_t = \sum_{i=1}^m I(Y_i < t) \lambda^i(t). \tag{9}$$

Then, if minimal repair is done at each failure time on the respective component i, the corresponding minimal repair counting process in $[0, t]$ is a Non-Homogenous Poisson Process (NHPP) with the Doob-Meyer decomposition (or Smooth Semimartingale (SSM) representation) given by

$$\widetilde{N}_t^i = \int_0^t \lambda^i(s) \, ds + \widetilde{M}_t^i, \quad \widetilde{M}_t^i \in \mathcal{M}_0^2. \tag{10}$$

Consequently, the expected number of minimal repairs on the component i in the interval $[0, t]$ is $E[\widetilde{N}_t^i] = \int_0^t \lambda^i(s) \, ds$.

Let $H_i(t)$ be the minimal repair discounted cost of component i at time t, a deterministic, continuous, decreasing, bounded, and integrable function on the interval $[0, t]$, such that $\int_0^t H_i(s) \lambda^i(s) \, ds < \infty, \forall \, 0 \le t < \infty$, and let $\widehat{B}_t^i = \sum_{j=1}^{\widetilde{N}_t^i} H_i(S_{ij})$ be the accumulated cost process by minimal repairs on component i in $[0, t]$, where S_{ij} is the time of the j-th minimal repair of component i and $S_{i1} = S_i$. Since the function $H_i(t)$ is

predictable and bounded in the interval $[0, t]$, and $E[\widetilde{N}_t^i] = \int_0^t \lambda^i(s)\, ds < \infty$, then the process $\int_0^t H_i(s)\, d\widetilde{M}_s^i$ is a mean zero and square integrable $\mathcal{F}_t$-Martingale (Fleming & Harrington, 1991), whose $\mathcal{F}_t$-compensator is

$$B_t^i = \int_0^t H_i(s)\,\lambda^i(s)\, ds < \infty, \forall\, 0 \le t < \infty. \tag{11}$$

Suppose that only the critical component causing the system failure is minimally repaired, that is, consider the set $\mathcal{C}^i = \{\omega \in \Omega : S_i(\omega) > Y_i(\omega)\}$. Gonzalez & Bueno (2011) show that in $\mathcal{C}^i$,

i. The $\mathcal{F}_t$-compensator of $\widetilde{N}_t^i$ is the process

$$\widetilde{A}_t^{i*} = \int_0^t I\,(Y_i < s)\,\lambda^i(s)\, ds = \int_{Y_i}^t \lambda^i(s)\, ds < \infty, \quad \forall\, 0 \le t < \infty \ P\text{-a.s.} \tag{12}$$

ii. The $\mathcal{F}_t$-compensator of $\widehat{B}_t^i$ is the process

$$\begin{aligned}
B_t^{i*} &= \int_0^t I\,(Y_i < s)\, H_i(s)\,\lambda^i(s)\, ds \\
&= \int_{Y_i}^t H_i(s)\,\lambda^i(s)\, ds < \infty, \quad \forall\, 0 \le t < \infty,\ P\text{-a.s.} \tag{13}
\end{aligned}$$

Based on the previous results, which apply to each system component, the minimal repair counting process and their corresponding cost processes for a coherent system are as defined next.

Definition 1. *(Gonzalez & Bueno, 2011) For each $\omega \in \Omega$, the set of components which survive its critical level is defined as*

$$\mathcal{C}^\Phi(\omega) = \{i \in \{1, \dots, m\} : S_i(\omega) > Y_i(\omega)\}. \tag{14}$$

For each $i = 1, \dots, m$ define $\quad C^i(\omega) = \begin{cases} 1 & \text{if } i \in \mathcal{C}^\Phi(\omega) \\ 0 & \text{otherwise} \end{cases}.$

Then, the minimal repair counting process of the coherent system in $[0, t]$ is $\widetilde{N}_t(\omega) = \sum\limits_{i=1}^m C^i(\omega)\,\widetilde{N}_t^i(\omega)$ and the warranty cost process is given by $\widehat{B}_t(\omega) = \sum\limits_{i=1}^m C^i(\omega)\,\widehat{B}_t^i(\omega)$, whose compensator process is $B_t(\omega) = \sum\limits_{i=1}^m C^i(\omega)\, B_t^i(\omega)$.

The $\mathcal{F}_t$-Martingale estimator of process $B_t^i(\omega)$, $i = 1, \dots, m$, in $\mathcal{C}^i$, is the process $\widehat{B}_t^i(\omega) = \int_0^t H_i(s)\, d\widetilde{N}_s^i(\omega)$ (Gonzalez & Bueno, 2011). Then, from Definition 1 and equation (13), the expected cost of minimal repairs carried out over the system in interval $[0, t]$ is

$$B^*(t) = E\left[\widehat{B}_t\right] = \sum_{i=1}^m B^{i*}(t), \tag{15}$$

with

$$B^{i*}(t) = P(S_i > Y_i)\, E\left[\left.\int_{Y_i}^t H_i(s)\,\lambda^i(s)\, ds\,\right|\, S_i > Y_i\right]. \tag{16}$$

Based on the results above, next section gives it is established an estimation method for $B^*(t)$ based on a random sample of n identical systems (or n independent copies of the process).

2.2. Estimation Based on a Sample of n Identical Systems

Consider n independent copies of the process $\left(\widehat{B}_t^i, C^i, i = 1, \ldots, m\right)_{t \geq 0}$ which are denoted by $\left(\widehat{B}_t^{i(j)}, C^{i(j)}, i = 1, \ldots, m, j = \right.$

For each j, let $\mathcal{C}^{\Phi(j)}$ be the set of critical components for the system j defined in (14), then the minimal repair costs for the j-th system is

$$\widehat{B}_t^{(j)} = \sum_{i=1}^{m} C^{i(j)} \int_0^t H_i(s) \, d\widetilde{N}_s^{i(j)}, \tag{17}$$

whose compensator process is

$$B_t^{(j)} = \sum_{i=1}^{m} C^{i(j)} \int_{Y_i^{(j)}}^t H_i(s) \lambda^i(s) \, ds. \tag{18}$$

From the n independent copies, the following mean processes are obtained,

$$\overline{\widehat{B}}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \widehat{B}_t^{(j)} \quad \text{y} \quad \overline{B}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} B_t^{(j)}.$$

The following are results for the mean processes defined above.

Theorem 1. *(Gonzalez & Bueno, 2011)*

i. $\overline{\widehat{B}}_t^{(n)}$ *is a consistent and unbiased estimator for $B^*(t)$.*

ii. *A consistent and unbiased estimator for* $Var\left[\overline{\widehat{B}}_t^{(n)}\right]$ *is* $\widehat{Var}\left[\overline{\widehat{B}}_t^{(n)}\right] = \sum_{i=1}^{m} \varrho_t^{2i(n)}/n.$

iii. *An approximate $100(1-\gamma)\%$ confidence interval for $B^*(t)$, is*

$$\overline{\widehat{B}}_t^{(n)} \pm Z_{\gamma/2} \sqrt{\frac{\sum\limits_{i=1}^{m} \varrho_t^{2i(n)}}{n}}, \tag{19}$$

where $Z_{\gamma/2}$ is the $(1 - \gamma/2)$ quantile of the standard normal distribution and

$$\varrho_t^{2\,i(n)} = \left(\frac{n}{n-1}\right) \frac{1}{n} \sum_{j=1}^{n} C^{i(j)} \left[\left(\widehat{B}_t^{i(j)} - B^{i*}(t)\right)^2 - \left(\overline{\widehat{B}}_t^{i(n)} - B^{i*}(t)\right)^2\right],$$

Despite having a set of approximately $100(1-\gamma)\%$ level pointwise confidence limits in $[0,t]$, given by (19), they do not form $100(1-\gamma)\%$ level simultaneous confidence bands. The following section introduces a general definition of simultaneous confidence bands for random functions.

2.3. Simultaneous Confidence Bands for Random Functions

Suppose the goal is to estimate and bound a function $f(t)$ in the interval $[0,t]$. That is, given a coverage probability of $(1-\gamma)$, we want to find two random functions $b_1(s)$ and $b_2(s)$ with the property

$$P\left[b_1(t) \leq f(t) \leq b_2(t), \, \forall\, s \in [0,t]\right] \approx 1 - \gamma, \tag{20}$$

Except for functions having a very simple structure, there are no simultaneous confidence bands with an exact $(1-\gamma)$ coverage probability (Knowles, 1988).

Let $\left(\widehat{f}_t^{(n)}\right)_{t \geq 0}$ be an estimator of the function $f(t)$, based on a random sample of size n, then the weak convergence of processes with the form $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ provides a general method for calculating simultaneous confidence bands for the function $f(t)$ (Fleming & Harrington, 1991). When $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ converges in distribution ($\xrightarrow{w}$) on interval $[0,t]$ to a limit process Q, the Continuous Mapping Theorem implies that

$$\sup_{0 \leq s \leq t} \sqrt{n} \left|\widehat{f}_s^{(n)} - f(s)\right| \xrightarrow{w} \sup_{0 \leq s \leq t} |Q(s)|. \tag{21}$$

If $q_\gamma(t)$ satisfies

$$P\left(\sup_{0 \leq s \leq t} |Q(s)| \leq q_\gamma(t)\right) \approx 1 - \gamma, \tag{22}$$

where $q_\gamma(t)$ is the $(1-\gamma)$ quantile in the distribution of $\sup_{0 \leq s \leq t} |Q(s)|$, then, asymptotically,

$$P\left(\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s^{(n)} - f(s)\right| \leq q_\gamma(t)\right) \approx 1 - \gamma. \tag{23}$$

Then, the construction of simultaneous confidence bands is based on finding $q_\gamma(t)$ which satisfies the desirable coverage probability on the interval $[0, t]$.

Simultaneous confidence bands based on $\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s^{(n)} - f(s)\right|$, will be useful only when sufficient conditions for the convergence of $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ on reasonable intervals $[0, t]$ are not too restrictive, when $q_\gamma(t)$ is easy to calculate, and when the resulting bands have appealing properties (Fleming & Harrington, 1991). Even when the general conditions for weak convergence could be fulfilled, calculating $q_\gamma(t)$ requires determining the limit process $\sup_{0 \leq s \leq t} |Q(s)|$ to which the process $\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s - f(s)\right|$ converges, and this is not easy when only a sample of n systems is available.

The following section presents the weak approach of processes introduced by Belyaev (2000) and Belyaev & Seleznjev (2000) as an extension of weak convergence of processes, which justifies the use of resampling in the approximation of asymptotic distributions.

2.4. Weakly Approaching Distributions

Let $\{\mathcal{L}(U_n)\}_{n \geq 1}$ and $\{\mathcal{L}(V_n)\}_{n \geq 1}$ be two sequences of distributions of random variables U_n and V_n which have values on $\mathcal{H}_n$, a metric space with metric d_n, and let $\mathscr{C}_b(\mathcal{H}_n, d_n)$ be the set of all bounded real-valued continuous functions on $\mathcal{H}_n$.

Definition 2. *(Belyaev, 2000) The distributions $\{\mathcal{L}(U_n)\}_{n \geq 1}$ weakly approach $\{\mathcal{L}(U_n)\}_{n \geq 1}$, denoted by $\mathcal{L}(U_n) \overset{wa}{\longleftrightarrow} \mathcal{L}(V_n)$, $n \to \infty$, if for all $h = h(\cdot) \in \mathscr{C}_b(\mathcal{H}_n, d_n)$ it holds that*

$$E[h(U_n)] - E[h(V_n)] \to 0, \ n \to \infty. \tag{24}$$

Here after we make the assumption that the random variable $W_n \in \mathcal{W}_n$ (a metric space) is defined on the same probability space of U_n. Also suppose that the regular conditional distribution $\mathcal{L}(U_n | W_n)$ exists.

Definition 3. *(Belyaev, 2000) The Random distributions $\{\mathcal{L}(U_n | W_n)\}_{n \geq 1}$ weakly approach $\{\mathcal{L}(V_n)\}_{n \geq 1}$ in probability, denoted by $\mathcal{L}(U_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n)$, $n \to \infty$, if for all $h \in \mathscr{C}_b(\mathcal{H}_n, d_n)$ the condition $E[h(U_n) | W_n] - E[h(V_n)] \overset{P}{\longrightarrow} 0$, $n \to \infty$ is satisfied. Here $\overset{P}{\longrightarrow}$ denotes convergence in probability.*

Lemma 1. *(Belyaev, 2000) Let U_n, W_n and V_n be as defined before. Suppose that $\mathcal{L}(U_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n)$ and let Z_n be an $\mathcal{H}_n$-valued random variable defined on the same probability space of U_n, such that $Z_n \overset{P}{\underset{w}{\longrightarrow}} 0, n \to \infty$. Then,*

$$\mathcal{L}(U_n + Z_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n), \ n \to \infty. \tag{25}$$

The notion of weakly approaching establishes a variant of Lyapunov's Central Limit Theorem (CLT) for $\mathbb{R}^k$, as follows. Let $\mathbb{U}_n = \{\mathbf{U}_{1n}, \ldots, \mathbf{U}_{nn}\}$, $n = 1, 2, \ldots$, be a triangular scheme of independent vector-valued random variables, where for each n, $\mathbf{U}_{in} = (U_{1in}, \ldots, U_{kin})^T$. Let $\mathbb{C}_{in}$, $i = 1, \ldots, n$ be the covariance matrix between the k variables of $\mathbf{U}_{in}$ and define $\mathbf{U}_{\cdot n} = \sum_{i=1}^n \mathbf{U}_{in}$, $\boldsymbol{\mu}_{\cdot n} = \sum_{i=1}^n E(\mathbf{U}_{in})$ and $\overline{\mathbb{C}}_{\cdot n} = (1/n) \sum_{i=1}^n \mathbb{C}_{in}$.

Theorem 2. *CLT in $\mathbb{R}^k$ (Belyaev, 2000) Suppose that for some constants $\delta > 0$ and $c = c(2 + \delta) < \infty$, $E|\sqrt{n} U_{hin}|^{2+\delta} \leq c$ for all $(i, n) \in \mathcal{T}$. Then it holds that*

$$\mathcal{L}\left(\sqrt{n}\left(\mathbf{U}_{\cdot n} - \boldsymbol{\mu}_{\cdot n}\right)\right) \overset{wa}{\longleftrightarrow} N_k\left(\mathbf{0}_k, \overline{\mathbb{C}}_{\cdot n}\right), n \to \infty, \tag{26}$$

where $N_k\left(\mathbf{0}_k, \overline{\mathbb{C}}_{\cdot n}\right)$ is the k-dimensional normal distribution with mean $\mathbf{0}_k$ and the covariance matrix $\overline{\mathbb{C}}_{\cdot n}$.

Now consider a triangular scheme $\mathbb{U}_n = \{\mathbf{U}_{1n}, \ldots, \mathbf{U}_{nn}\}$ of independent vector-valued random variables with $\mathbf{U}_{in} \in \mathbb{R}^k$, $(i, n) \in \mathcal{T}$, where $\mathcal{T} = \{(i', n') \, i' = 1, \ldots, n', n' = 1, 2, \ldots\}$. Let $J_{1n}^\star, \ldots, J_{nn}^\star$ be the indexes of a resample from $\mathbb{U}_n$, indicating which of the observations $\mathbf{U}_{in}$ is chosen as the i-th element in the resample. Let $N_{hn}^\star = \sum_{i=1}^n I\left(J_{in}^\star = h\right)$ be the number of times that the observation h is drawn in the resample and define the vector-valued random variable

$$\mathbf{U}_{\cdot n}^{\star 0} := \sum_{i=1}^n \left(N_{in}^\star - 1\right) \mathbf{U}_{in}.$$

Theorem 3. *CLRT in $\mathbb{R}^k$ (Belyaev, 2000) Suppose the assumptions of Theorem 2 are fulfilled and that $E\left[U_{hin}\right] = \mu_{hn}$, that is, the expectation does not dependent of i. Then, it holds that*

$$\mathcal{L}\left(\boldsymbol{U}_{\cdot n}^{\star 0} \middle| \mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(\boldsymbol{U}_{\cdot n} - \boldsymbol{\mu}_{\cdot n}\right), \ n \to \infty. \tag{27}$$

In general, the resampling process consists of simulating G copies $\{J_{1n}^{\star r}, \ldots, J_{nn}^{\star r}\}$, $r = 1, \ldots, G$, (which are used to approximate $\mathcal{L}\left(\mathbf{U}_{\cdot n}^{\star 0} \middle| \mathbb{U}_n\right)$). Then, $\mathbf{U}_{\cdot n}^{\star 0 r} = \sum_{i=1}^n \left(N_{in}^{\star r} - 1\right) \mathbf{u}_{in}$ is obtained for $r = 1, \ldots, G$, where $\mathbf{u}_{in}$ is the observed value of $\mathbf{U}_{in}$. So, for any Borel set $A \subset \mathbb{R}^k$ it has that

$$\frac{1}{G} \sum_{r=1}^G I\left(\mathbf{U}_{\cdot n}^{\star 0 r} \in A\right) \overset{P}{\longrightarrow} P\left(\mathbf{U}_{\cdot n}^{\star 0} \in A \middle| \mathbb{U}_n\right), \ G \to \infty. \tag{28}$$

Belyaev (2000) shows that to assess the accuracy of some nonparametric estimators, it is necessary to consider more general spaces than $\mathbb{R}^k$. For instance, for many non-parametric statistical models it is necessary to consider the Skorokhod space $D\left[0, t\right], t > 0$, of the so-called *cadlag* functions. That is, the set of all real functions $v\left(\cdot\right)$ defined in $[0, t]$ such that for all $s \in (0, t)$ there are limit values $v\left(s \pm 0\right) = \lim_{h \downarrow 0} v\left(s \pm h\right)$ and $v\left(s\right) = v\left(s + 0\right)$.

Theorem 4. *CLRT in Skorokhod space (Belyaev, 2000). For the $\mathbb{U}_n = \{U_{1n}\left(t\right), \ldots, U_{nn}\left(t\right)\}$ triangular scheme of $D[0, t]$-valued random variables which are independent for each $n = 1, 2, \ldots$. Let $U_{\cdot n}\left(t\right) = \sum_{i=1}^n U_{in}\left(t\right)$. Suppose that $E\left[U_{in}\left(s\right)\right] = \mu_n\left(s\right), i = 1, \ldots, n, s \in [0, t]$ and that there are positive constants c_1, c_2, c_3 and $\delta > 0$ such that for every $0 \le t_1 \le s \le t_2 \le t$.*

C-1. $nE[\left(U_{jn}\left(t_2\right) - U_{jn}\left(t_1\right)\right)^2] \le c_1 \left|t_2 - t_1\right|^{(1+\delta)/2}$,

C-2. $n^2 E[\left(U_{jn}\left(s\right) - U_{jn}\left(t_1\right)\right)^2 \left(U_{jn}\left(t_2\right) - U_{jn}\left(s\right)\right)^2] \le c_2 \left|t_2 - t_1\right|^{1+\delta}$, and

C-3. $E\left[\left|\sqrt{n}U_{jn}\left(s\right)\right|^{2+\delta}\right] \le c_3$.

Then, it holds that

$$\mathcal{L}\left(\sum_{j=1}^n \left(N_{jn}^\star - 1\right) U_{jn}\left(\cdot\right) \middle| \mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}\left(\cdot\right) - \mu_n\left(\cdot\right)\right), \ n \to \infty. \tag{29}$$

3. Simultaneous Confidence Bands for the Expected Discounted Warranty Cost

In constructing simultaneous confidence bands for $B^*\left(t\right)$ it is important to assess the distribution of the unobservable processes $\sqrt{n}\left(\overline{\widehat{B}}_t^{(n)} - B^*\left(t\right)\right)$. Theorem 4 and the use of resampling allow approximating relevant distribution when information about an initial sample of n systems is available. For this purpose, define

$$U_{\cdot n}\left(t\right) = \sqrt{n}\left(\overline{\widehat{B}}_t^{(n)} - B^*\left(t\right)\right) = \sum_{j=1}^n \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - B^*\left(t\right)\right) = \sum_{j=1}^n U_{jn}\left(t\right) \tag{30}$$

where,

$$U_{jn}\left(t\right) = \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - B^*\left(t\right)\right), \tag{31}$$

are stochastic processes in $D[0, t]$, which can be arranged in a triangular scheme $\mathbb{U}_n$.

To use the CLRT in $D[0, t]$, it must be verified that the processes $U_{jn}(t)$ satisfy the conditions *C-1*, *C-2* and *C-3* established in Theorem 4, which is summarized in the following theorem (see proof in Lopera (2014))

Theorem 5. *Let $U_{jn}(t)$ be as defined in (31). Then for every $0 \leq t_1 \leq s \leq t_2 \leq t < \infty$, there exist positive constants c_1, c_2, c_3 and $\delta > 0$, such that*

C-1. $nE[(U_{jn}(t_2) - U_{jn}(t_1))^2] \leq c_1 |t_2 - t_1|^{(1+\delta)/2}$,

C-2. $n^2 E[(U_{jn}(s) - U_{jn}(t_1))^2 (U_{jn}(t_2) - U_{jn}(s))^2] \leq c_2 |t_2 - t_1|^{1+\delta}$, *and*

C-3. $E\left[|\sqrt{n}\, U_{jn}(s)|^{2+\delta}\right] \leq c_3$.

The following corollary formalizes the application of the CLRT to warranty cost processes.

Corollary 1. *Consider a sample of n independent copies of $\left(\widehat{B}_t^i, C^i, i = 1, \ldots, m\right)_{t \geq 0}$ and the triangular scheme $\mathbb{U}_n(t) = \{U_{n1}(t), \ldots, U_{nn}(t)\}$ with $U_{jn}(t)$ given in (31). Then, for the process*

$$U_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n} \left(N_{jn}^{\star} - 1\right) U_{jn}(t), \tag{32}$$

one can show that

$$\mathcal{L}\left(U_{\cdot n}^{\star 0}(t) \big| \mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}(t)\right), \quad n \to \infty. \tag{33}$$

Proof. By applying Theorem 5 over the process $U_{jn}(t)$, the necessary conditions for Theorem 4 are obtained, from whose application the result is obtained. $\square$

In practice the processes $U_{jn}(t)$ are unknown and they need to be estimated. The following section uses estimations of the processes $U_{jn}(t)$ to construct simultaneous confidence bands based on resampling.

3.1. A Proposal for Simultaneous Confidence Bands for the Expected Warranty Cost

The unobservable $U_{jn}(t)$ can be rewritten as follows:

$$U_{jn}(t) = \widehat{U}_{jn}(t) + \widetilde{U}_{jn}(t), \tag{34}$$

where

$$\widehat{U}_{jn}(t) = \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - \overline{\widehat{B}}_t^{(n)}\right) \quad \text{y} \quad \widetilde{U}_{jn}(t) = \frac{1}{\sqrt{n}}\left(\overline{\widehat{B}}_t^{(n)} - B^*(t)\right).$$

The following result establishes that the weakly approaching of the process $U_{\cdot n}^{\star 0}(t)$, given in (32), is kept in an estimated version of itself.

Corollary 2. *The process $\widehat{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)\widehat{U}_{jn}(t)$ has the property that*

$$\mathcal{L}\left(\widehat{U}_{\cdot n}^{\star 0}(t)\big| \mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}(t)\right), \quad n \to \infty. \tag{35}$$

Proof. By using (34) in (32), the identity

$$U_{\cdot n}^{\star 0}(t) = \widehat{U}_{\cdot n}^{\star 0}(t) + \widetilde{U}_{\cdot n}^{\star 0}(t), \tag{36}$$

is obtained, where $\widehat{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)\widehat{U}_{jn}(t)$ and $\widetilde{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)\widetilde{U}_{jn}(t)$.

Notice that $\widetilde{U}_{\cdot n}^{\star 0}(t) \equiv 0$ (since $\widetilde{U}_{\cdot n}^{\star 0}(t)$ does not dependent of j by definition and $\sum_{j=1}^{n} N_{jn}^{\star} = n$). Then, $\widetilde{U}_{\cdot n}^{\star 0}(t) \overset{P}{\longrightarrow} 0$, therefore by using Lemma 1 the result is obtained. $\square$

The latter allows proposing simultaneous confidence bands for the expected discounted warranty cost in coherent systems under physical minimal repair.

Theorem 6. *An approximate* $100\,(1-\gamma)\,\%$ *simultaneous confidence band for* $B^{*}\,(t)$, *the expected discounted warranty cost process for a coherent system under physical minimal repair on the interval* $[0,t]$, *is*

$$\overline{\overline{B}}_{t}^{(n)} \pm \frac{q_{\gamma}^{\star}\,(t)}{\sqrt{n}},\ 0 \leq s \leq t, \tag{37}$$

where, $q_{\gamma}^{\star}\,(t)$ *is the* $(1-\gamma)$ *quantile of the empirical distribution of* $\sup_{0 \leq s \leq t}\left|\widehat{U}_{.n}^{\star 0}\,(s)\right|$.

Proof. Gonzalez & Bueno (2011) showed that the process $U_{.n}\,(t)$ weakly converges in the space $D\,[0,t]$ to a Gaussian stochastic process denoted by $Q\,(t)$ which implies

$$\sup_{0 \leq s \leq t} |U_{.n}\,(s)| \xrightarrow{w} \sup_{0 \leq s \leq t} |Q\,(s)|. \tag{38}$$

By following the definition of simultaneous confidence bands introduced in Section and from (38) the idea is finding $q_{\gamma}\,(t)$ such that:

$$P\left(\sup_{0 \leq s \leq t} |Q\,(s)| \leq q_{\gamma}\,(t)\right) \approx 1-\gamma, \tag{39}$$

where the process $\sup_{0 \leq s \leq t} |Q\,(s)|$, to which the process $\sup_{0 \leq s \leq t} |U_{.n}\,(s)|$ weakly converges is unknown. Using Corollary 1

$$\mathcal{L}\left(U_{.n}^{\star 0}\,(t)\,\middle|\,\mathbb{U}_{n}\right) \xrightarrow{wa(P)} \mathcal{L}\left(U_{.n}\,(t)\right),\ n \to \infty.$$

Thus, instead of finding $q_{\gamma}\,(t)$ satisfying (39), the purpose is obtaining $q_{\gamma}^{\star}\,(t)$ such that:

$$P\left(\sup_{0 \leq s \leq t}\left|U_{.n}^{\star 0}\,(s)\right| \leq q_{\gamma}^{\star}\,(t)\right) \approx 1-\gamma. \tag{40}$$

This is equivalent, using Corollary 2, to find the value of $q_{\gamma}^{\star}\,(t)$ such that

$$P\left(\sup_{0 \leq s \leq t}\left|\widehat{U}_{.n}^{\star 0}\,(s)\right| \leq q_{\gamma}^{\star}\,(t)\right) \approx 1-\gamma. \tag{41}$$

Thus, $q_{\gamma}^{\star}\,(t)$ can be chosen as the $(1-\gamma)$ quantile of the empirical distribution based on resampling from $\sup_{0 \leq s \leq t}\left|\widehat{U}_{.n}^{\star 0}\,(s)\right|$. Thus, considering that

$$\sup_{0 \leq s \leq t}\left|\widehat{U}_{.n}^{\star 0}\,(s)\right| = \lim_{M \to \infty}\ \max_{t_{j} \in \mathcal{A}}\left|\widehat{U}_{.n}^{\star 0}\,(t_{j})\right|, \tag{42}$$

where $\mathcal{A} = \{t_{0} = 0 \leq t_{1} \leq \cdots \leq t_{M-1} \leq t_{M} = t\}$ is a partition of interval $[0,t]$, then, with a value of M large enough, it holds that

$$P\left(\overline{\overline{B}}_{t}^{(n)} - \frac{q_{\gamma}^{\star}\,(t)}{\sqrt{n}} \leq B^{*}\,(s) \leq \overline{\overline{B}}_{t}^{(n)} + \frac{q_{\gamma}^{\star}\,(t)}{\sqrt{n}} :\ 0 \leq s \leq t\right) \approx 1-\gamma, \tag{43}$$

such that $\overline{\overline{B}}_{t}^{(n)} \pm q_{\gamma}^{\star}\,(t)/\sqrt{n},\ 0 \leq s \leq t$, is an approximate $(1-\gamma)$ level simultaneous confidence band for $B^{*}\,(t)$ within interval $[0,t]$. $\square$

4. Simulation Study
This section uses simulation to assesses the properties of the proposed simultaneous confidence bands. The simulation study considers different scenarios which depend on factors that may affect the performance of the bands established in Theorem 6.

4.1. Simulation Factors and Parameters
The following are the factors considered in the simulation study:

- **System type.** A three-component parallel system and a 2-out-of-4 components system are considered. To avoid confusion, both systems are treated as k-out-of-m systems, which are denoted by: a) $\Phi_t^{1:3}$ for the three-components parallel system or 1-out-of-3 components system, and b) $\Phi_t^{2:4}$ for the 2-out-of-4 components system.

- **Number of systems under warranty.** This is denoted by n and corresponds to the number of independent and identical copies of the repair/cost process used for constructing simultaneous confidence bands. The levels considered are $n = 10, 30, 50, 100, 500$ and 1000.

- **Discount function.** Denoted by $H_i(t)$ describes the consumer share of physical minimal repair costs for the system during the W warranty term. $H_i(t) = c_i e^{-t}$, $c_i \left(1 - tW^{-1}\right) e^{-t}$, $i = 1, \ldots, m$ were used.

- **Number of resamples.** This is denoted by G. The levels of G are 500, 1000, 5000 and 10000.

- **Partition size.** This is denoted by M and it determines how thin is the partition of the warranty period, for the approximation of the supreme of the limit process given in (38). The levels of M are 100, 500 and 1000.

Table 1 summarizes the levels considered in the simulation factors.

Table 1: Simulation Factors and Their Levels

Factor	Levels
$\Phi_t^{k:m}$	$\Phi_t^{1:3},\ \Phi_t^{2:4}$
n	$10, 30, 50, 100, 500, 1000$
$H_i(t)$	$c_i e^{-t},\ c_i\left(1 - tW^{-1}\right)e^{-t}$
G	$500, 1000, 5000, 10000$
M	$100, 500, 1000$

The following are fixed values or simulation parameters:

- **Component failure distributions.** For the systems considered ($\Phi_t^{1:3}$ and $\Phi_t^{2:4}$) in each component i, whose associated lifetime is denoted S_i, the respective cumulative distribution function F_i is considered, thus: $F_i(t) = \text{Weibull}(\eta = 1.0,\ \beta = 1.5)$, $i = 1, 2$; $F_i(t) = \text{Weibull}(\eta = 2.0,\ \beta = 2.0)$, $i = 3, 4$.

- **Warranty term.** W denotes the time period within which the system is under warranty. The simulation uses $W = 5$. This can be interpreted as representing five years or five thousand use cycles.

- **Minimal repair cost by component.** It corresponds to the minimal repair value in the i-th component and is denoted c_i. For this study, $c_1 = c_2 = 3$ and $c_3 = c_4 = 5$ were considered.

- **Nominal coverage probability.** Denoted by $(1 - \gamma)$. It specifies the expected probability that the true mean cost function is bounded. A value of $(1 - \gamma) = 0.95$ was considered.

- **Number of simulations.** This is denoted by N. A total of $N = 10000$ simulations were used.

The Weibull distribution for component lifetime was chosen for its frequent use in industrial reliability. Besides, the values of Weibull distribution parameters set for each component ensure the distributions have increasing failure rates, and a record of failures with at least one event during the established warranty term. The constrains $F_1 \equiv F_2$ and $F_3 \equiv F_4$ were used to simplify the simulation scenarios.

Table 2 summarizes the fixed values for the parameters considered in the simulation study.

Table 2: Fixed Values of Simulation Parameters

Parameter	Fixed value
$F_i(t),\ i = 1, 2$	Weibull $(\eta = 1.0, \beta = 1.5)$
$F_i(t),\ i = 3, 4$	Weibull $(\eta = 2.0, \beta = 2.0)$
W	5
c_i	$c_1 = c_2 = 3,\ c_3 = c_4 = 5$
$(1 - \gamma)$	0.95
N	10000

The purpose, at this point, is to assess the performance of the simultaneous confidence band proposed in each scenario, by estimating the coverage probabilities.

4.2. Coverage Probabilities

Let $\text{SCB}_t^{(n)}$ be a simultaneous confidence band for the function $B^*(t)$ in $[0, t]$, based on a sample of n systems, then the coverage probability (CP) for $\text{SCB}_t^{(n)}$ is defined as:

$$\text{CP} = P\left(B^*(t) \in \text{SCB}_t^{(n)}\right). \tag{44}$$

If simultaneous confidence band $\text{SCB}_t^{(n)}$ has a $(1 - \gamma)$ level, then $\text{CP} \approx 1 - \gamma$.

The following is the procedure followed during the simulations:

i. For each scenario, generate N simulations of n systems under warranty.

ii. For each simulation:

    (a) Observe the component failure processes in $[0, W]$, record $C^{i(j)}$, $\widehat{B}_t^{i(j)}$, $\widehat{B}_t^{(j)}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and calculate $\overline{\widehat{B}}_t^{(n)}$.

    (b) Obtain G resamples of size n say $\{J_{1n}^{\star r}, \ldots, J_{nn}^{\star r}\}$, $r = 1, \ldots, G$. Calculate $\max\limits_{t_j \in \mathcal{A}} \left| \widehat{U}_{.n}^{\star 0}(t_j) \right|$ for each r, where $\mathcal{A}$ is a partition of size M of $[0, W]$.

    (c) Obtain $q_\gamma^\star(t)$, the $(1 - \gamma)$ quantile of the estimated approximate distribution for $\sup_{0 \leq s \leq t} \left| \widehat{U}_{.n}^{\star 0}(s) \right|$.

    (d) Using (37) and the information on n systems, obtain a $(1 - \gamma)$ level simultaneous confidence band for $B^*(t)$ ($\mathrm{SCB}_t^{l(n)}$, $l = 1, \ldots, N$).

iii. For each scenario, calculate the actual coverage probability for the simultaneous confidence band,

$$\widehat{\mathrm{CP}} = \frac{1}{N} \sum_{l=1}^{N} I\left(B^*(t) \in \mathrm{SCB}_t^{l(n)} \right), \tag{45}$$

where the indicator variables determine if the function $B^*(t)$ is totally contained within the resulting bands in each simulation.

Since the function of the expected cost $B^*(t)$ is unknown, it is approximated with $\overline{\overline{B}}_t^{(n)}$ for a sample of 100000 systems. Figure 1 shows the functions $B^*(t)$ for coherent systems and discount functions under study.

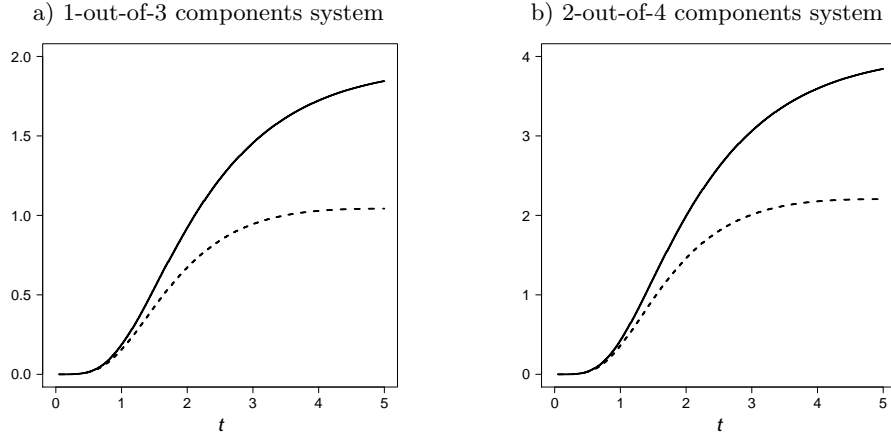a) 1-out-of-3 components system   b) 2-out-of-4 components system

Figure 1: Approximate $B^*(t)$ for systems under study. The solid curve is calculated for $H_i(t) = c_i \exp(-t)$ and the dashed cure is calculated for $H_i(t) = c_i \left(1 - tW^{-1} \right) \exp(-t)$.

Then, the actual coverage probability for the proposed simultaneous confidence bands is obtained from (45). The following are the results of the simulation study for each of the coherent systems considered.

4.3. Actual Coverage Probabilities for the 1-out-of-3 Components System

For each value considered of the partition size M of the warranty period, Figure 2 shows the results for analyzing the effects of the size of resamples G and the size of sample n, over the actual coverage probabilities for both discount functions.
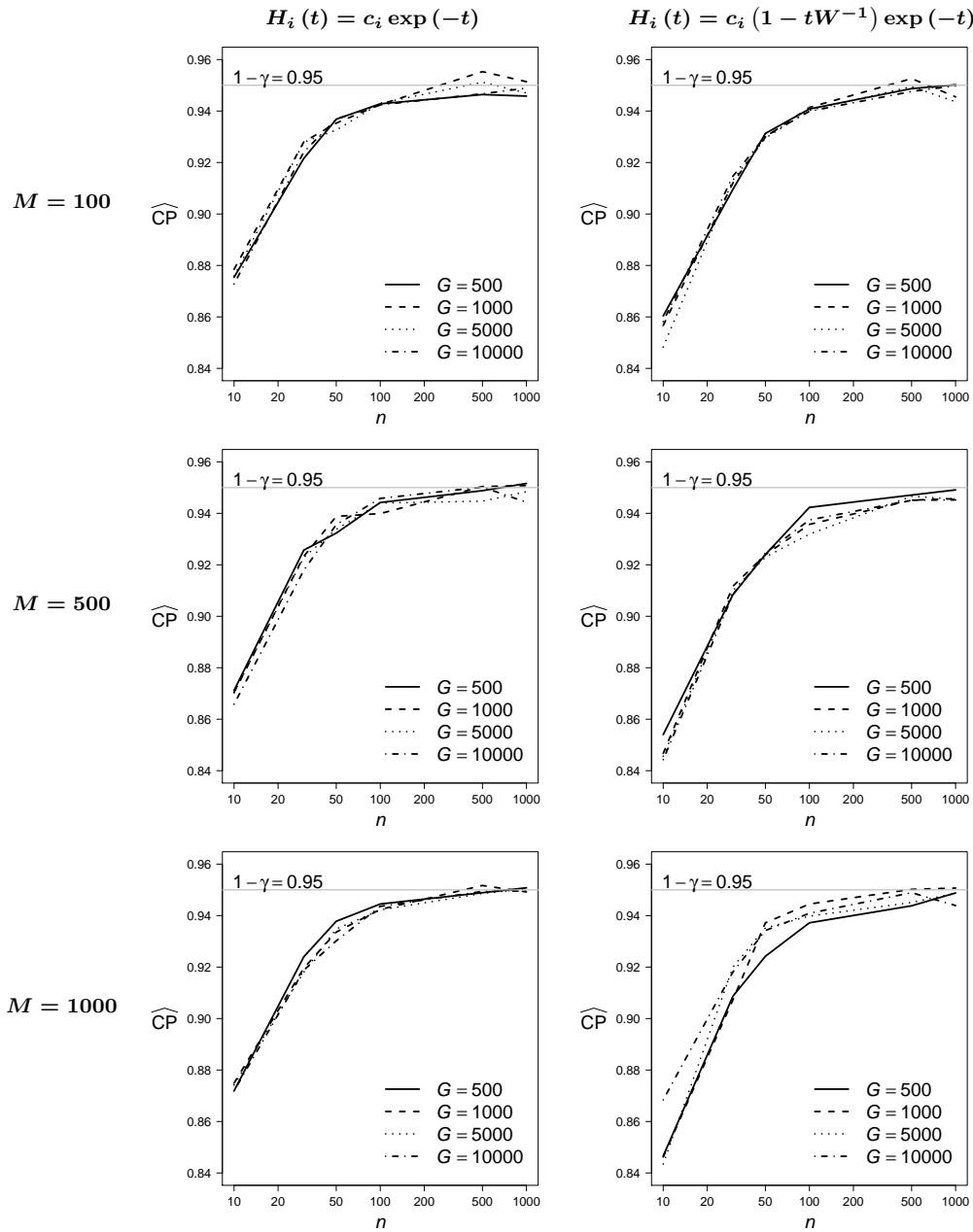
Figure 2: $\widehat{\mathrm{CP}}$ for $\mathrm{SCB}_t^{(n)}$, for the 1-out-of-3 components system, varying G for both discount functions.

Note that for the values of $M = 100$ and 500 considered for the partition size, there are just small differences between the $\widehat{\mathrm{CP}}$ curves for the resample sizes G studied in each discounted cost model considered. This suggests that for this study the resample size G does not affect the behavior of the actual coverage probabilities of the proposed bands. Nevertheless when dealing with fine partitions $M = 1000$, the actual coverage probabilities increase because the larger number of resamples. Figure 2 shows that the differences in actual coverage probabilities achieved by the two discount functions decrease when the number of systems increases, reaching values close to the nominal level of $(1 - \gamma)$ used in the simulation.

4.4. Actual Coverage Probabilities for the 2-out-of-4 Components System

Figure 3 shows the effect of the resample size G over the actual coverage probabilities under different sample sizes, for both discount functions. Figure 3 shows that, similar to, the 1-out-of-3 system, when the number
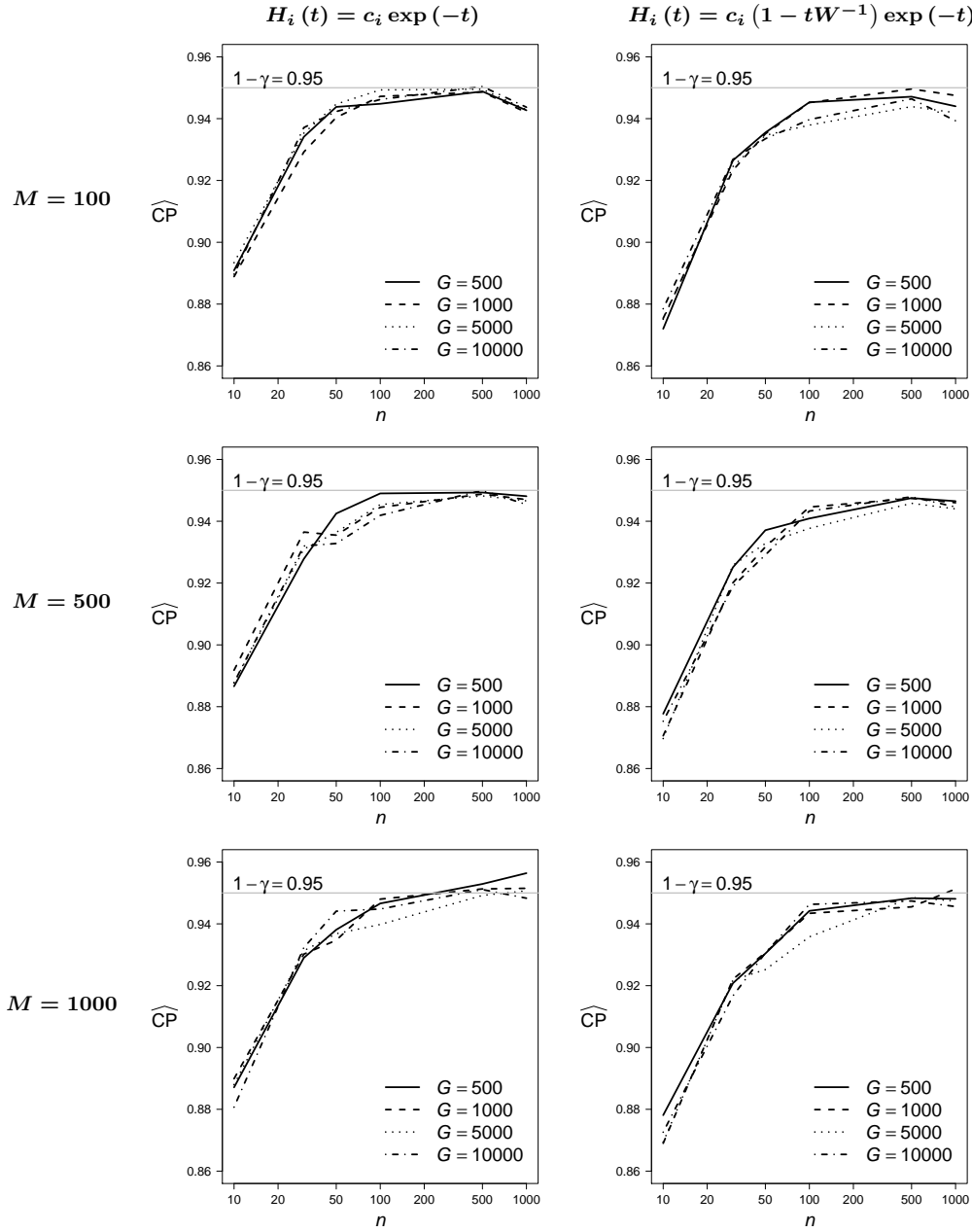

Figure 3: $\widehat{\mathrm{CP}}$ for $\mathrm{SCB}_t^{(n)}$, for the 1-out-of-3 components system, varying G for both discount functions.

of systems under warranty increases, the actual coverage probabilities of the simultaneous confidence bands increase toward the confidence level $(1 - \gamma)$. At each level of the partition size M, the smaller coverage probabilities suggest an improvement when the resample size G increases. It is worth noting that there are large differences between the coverage probabilities for both discount functions when the number of systems is smaller or equal than 100. But similar values to the nominal $(1 - \gamma)$ level are achieved in both discount functions when the number of systems is greater than 100.

5. Conclusions

For some statistical models, assessment of the precision of the statistical inferences may be carried out by means of intensive computer methods (Efron, 1979; Davison & Hinkley, 1997; Belyaev, 2007). Resampling was efficiently used in this work to obtain simultaneous confidence bands, for the expected discounted warranty cost under physical minimal repair. This is a useful tool to assess the precision of the estimator, avoiding the complications of the asymptotic analysis of the related stochastic processes.

The proposed computation of the simultaneous confidence bands is valid in a wide range of models that satisfy with the general conditions identified in Section 2.

The proposed computation of the simultaneous confidence bands for the expected discounted warranty cost of coherent systems under minimal repair is easy to implement in current statistical package, since it only involves random sampling with replacement. Also, in the simulation scenarios studied in Section 4, reasonable actual coverage probabilities were obtained, particularly when it there was a number of systems under warranty greater than 100, a fine partition of the warranty term and a large number of resamples.

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