On the parameters estimation of the new Seasonal FISSAR model and simulations

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Abstract

In this paper we discuss on the methods of estimating the parameters of the Seasonal FISSAR model. First we implement the regression method based on the log-periodogram and the classical Whittle method for estimating the memory parameters. For estimating all model parameters simultaneously, innovation parameters and memory parameters, the maximum likelihood method, and the Whittle method based on the MCMC simulation are considered. We can show empirically the consistency and we investigate the asymptotic normality of the estimators by various results from simulation. The simulation results give the average values, the corresponding sample standard deviation (sd), the root mean square error (RMSE) and the mean absolute error (MAE) of the estimation procedures based on Monte Carlo replications.

Keywords: FISSAR model; long memory; regression method; Whittle method; MLE method.

1 Introduction

Statistics for spatially referenced data are becoming increasingly important as they are nowadays collected in large quantities in various areas of science. This requires specific probabilistic models and corresponding statistical methods. Boissy et al. [2005] had extended the long memory concept from times series to the spatial context and introduced the class of fractional spatial autoregressive model. Shitan [2008] used independently this model, called the Fractionally Integrated Separable Spatial Autoregressive (FISSAR) model to approximate the dynamics of spatial data when the autocorrelation function decays slowly.

Inference problems in spatial location or two-dimensional process have been studied by several authors. Hosking [1981], Yajima [1985] and Fox and Taqqu [1986] have discussed problems of asymptotic estimation based on one-dimensional fractionally differenced autoregressive process. Sethuraman and Basawa [1995] have studied the asymptotic properties of the maximum likelihood estimators in two-dimensional lattice. The authors considered models which permit dependence between time points as well as within the group of individuals at each time point.

Periodic and cyclical behaviours are present in many observations, and the Seasonal FISSAR model will have a lot of applications, including the modelling of temperatures, rainfalls when the data are collected during differend seasons for different locations. Cisse et al. [2016] incorporated Seasonal patterns into the FISSAR, thus introduced the Seasonal FISSAR model and discussed on the statistical properties of the model. In this paper, we investigate on estimating procedures of this model parameters. Spatial analysis includes a variety of techniques, many still in their early development, using different analytic approaches and applied in fields as diverse as econometrics techniques.

The paper is organized as follows. In the next Section, we compute the methods for estimating the model parameters. The methods are illustrated in Section 3 with an extensive simulation study and an empirical applications. Conclusions are given in Section 4.

2 Estimation Procedures

In this section, we discuss the methods of estimation of the memory parameters and examine the properties of the estimators through a simulation study. The fundamental concept of parameter estimation is to determine optimal

values of parameters for a numerical model that predicts dependent variable outputs of a function, process or phenomenon based on observations of independent variable inputs.
Let $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a sequence of spatial observations on two dimensional regular lattices. The Seasonal FISSAR process is defined as

$$\begin{aligned}(1-\phi_{10}B_1-\phi_{01}B_2+\phi_{10}\phi_{01}B_1B_2)\,(1-\psi_{10}B_1^s-\psi_{01}B_2^s+\psi_{10}\psi_{01}B_1^sB_2^s)\\ \times(1-B_1)^{d_1}(1-B_1^s)^{D_1}(1-B_2)^{d_2}(1-B_2^s)^{D_2}X_{ij}=\varepsilon_{ij}\end{aligned} \tag{1}$$

This equation (1) is equivalent to the following equations:

$$(1-\phi_{10}B_1)(1-\phi_{01}B_2)(1-\psi_{10}B_1^s)(1-\psi_{01}B_2^s)X_{ij} = W_{ij} \tag{2}$$
$$(1-B_1)^{d_1}(1-B_1^s)^{D_1}(1-B_2)^{d_2}(1-B_2^s)^{D_2}W_{ij} = \varepsilon_{ij} \tag{3}$$

where s is the seasonal period, B_1 and B_2 are the usual backward shift operators acting in the i^{th} and j^{th} direction, respectively. In addition, the parameters d_1 and d_2 are called memory parameters D_1 and D_2 are fractional parameters and $\{\varepsilon_{ij}\}_{i,j\in\mathbb{Z}_+}$ is a two-dimensional white noise process, mean zero and variance σ_ε^2. We denote $\{W_{ij}\}_{i,j\in\mathbb{Z}_+}$ the two-dimensional seasonal fractionally integrated white noise process.

On the estimation procedure of the parameters models in times series, a well-known procedure is the Whittle Method. It is known that under fairly general assumptions, it leads to consistent estimates and asymptotically normal. More in the Gaussian case, it's asymptotically efficient. Here we will present this method in spatial context and make a comparison with a regression method based on the log-periodogram that we will present in following.

2.1 Log-periodogram Regression Method

Probably the most commonly applied semi parametric estimator in time series is the log-periodogram regression estimator introduced by [Geweke and Porter-Hudak, 1983]. [Ghodsi and Shitan, 2009] provided a regression method of estimating long memory processes in spatial context and we briefly describe this method for estimating the parameters of the Seasonal FISSAR model.
Let

$$H_{ij}=(1-B_1)^{d_1}(1-B_1^s)^{D_1}(1-B_2)^{d_2}(1-B_2^s)^{D_2}X_{ij}, \tag{4}$$

and $f_X(\lambda_1,\lambda_2)$ the spectral density function of the process defined in (1). Then $f_X(\lambda_1,\lambda_2)$ can be written as:

$$f_X(\lambda_1,\lambda_2)=\left|1-\mathrm{e}^{-i\lambda_1}\right|^{-2d_1}\left|1-\mathrm{e}^{-i\lambda_2}\right|^{-2d_2}\left|1-\mathrm{e}^{-is\lambda_1}\right|^{-2D_1}\left|1-\mathrm{e}^{-is\lambda_2}\right|^{-2D_2}f_H(\lambda_1,\lambda_2) \tag{5}$$

where

$$f_H(\lambda_1,\lambda_2)=\frac{\sigma_\varepsilon^2}{4\pi^2\left|1-\phi_{10}\,\mathrm{e}^{-i\lambda_1}\right|^2\left|1-\phi_{01}\,\mathrm{e}^{-i\lambda_2}\right|^2\left|1-\psi_{10}\,\mathrm{e}^{-is\lambda_1}\right|^2\left|1-\psi_{01}\,\mathrm{e}^{-is\lambda_2}\right|^2}. \tag{6}$$

Taking the natural logarithm in (5), we obtain:

$$\begin{aligned}\log f_X(\lambda_1,\lambda_2) = {}& \log f_H(0,0)-d_1\log\left|1-\mathrm{e}^{-i\lambda_1}\right|^2-d_2\log\left|1-\mathrm{e}^{-i\lambda_2}\right|^2-D_1\log\left|1-\mathrm{e}^{-is\lambda_1}\right|^2\\ &-D_2\left|1-\mathrm{e}^{-is\lambda_2}\right|^2+\log\frac{f_H(\lambda_1,\lambda_2)}{f_H(0,0)}\end{aligned} \tag{7}$$

Replacing λ_1 by $\omega_{1,j}$ and λ_2 by $\omega_{2,j}$ in (7), we obtain

$$\begin{aligned}\log f_X(\omega_{1,j},\omega_{2,j}) = {}& \log f_H(0,0)-d_1\log\left|1-\mathrm{e}^{-i\omega_{1,j}}\right|^2-d_2\log\left|1-\mathrm{e}^{-i\omega_{2,j}}\right|^2-D_1\log\left|1-\mathrm{e}^{-is\omega_{1,j}}\right|^2\\ &-D_2\left|1-\mathrm{e}^{-is\omega_{2,j}}\right|^2+\log\frac{f_H(\omega_{1,j},\omega_{2,j})}{f_H(0,0)}\end{aligned} \tag{8}$$

Denote $N_1\times N_2$ the size of the regular rectangle when the process $\{X_{ij}\}$ is defined, and define

$$I(\lambda_1,\lambda_2)=\frac{1}{4\pi^2N_1N_2}\left|\sum_{k=1}^{N_1}\sum_{l=1}^{N_2}X_{k,l}e^{-i(k\lambda_1+l\lambda_2)}\right|^2 \tag{9}$$

to be the periodogram of the process. By adding $\log I\left(\omega_{1,j},\omega_{1,j}\right)$ in (8) , we obtain:

$$
\begin{aligned}
\log I\left(\omega_{1,j},\omega_{1,j}\right) &= \log f_H(0,0) - d_1 \log\left|1-\mathrm{e}^{-i\omega_{1,j}}\right|^2 - d_2\log\left|1-\mathrm{e}^{-i\omega_{2,j}}\right|^2 - D_1\log\left|1-\mathrm{e}^{-is\omega_{1,j}}\right|^2 \\
&\quad -D_2\left|1-\mathrm{e}^{-is\omega_{2,j}}\right|^2 + \log\frac{f_H(\omega_{1,j},\omega_{2,j})}{f_H(0,0)} + \log\frac{I\left(\omega_{1,j},\omega_{1,j}\right)}{f_X(\omega_{1,j},\omega_{2,j})} \qquad (10)
\end{aligned}
$$

If $\omega_{1,j}$ and $\omega_{1,j}$ are close to zero, then the term $\log\frac{f_H(\omega_{1,j},\omega_{2,j})}{f_H(0,0)}$ would be negligible when compared with the other terms of (10), and equation (10) reduces to:

$$
\begin{aligned}
\log I\left(\omega_{1,j},\omega_{1,j}\right) &\approx \log f_H(0,0) - d_1 \log\left|1-\mathrm{e}^{-i\omega_{1,j}}\right|^2 - d_2\log\left|1-\mathrm{e}^{-i\omega_{2,j}}\right|^2 - D_1\log\left|1-\mathrm{e}^{-is\omega_{1,j}}\right|^2 \\
&\quad -D_2\left|1-\mathrm{e}^{-is\omega_{2,j}}\right|^2 + \log\frac{I\left(\omega_{1,j},\omega_{1,j}\right)}{f_X(\omega_{1,j},\omega_{2,j})}. \qquad (11)
\end{aligned}
$$

Therefore we can write (11) as a multiple regression equation

$$X_j = \beta_0 + \beta_1 Z_{1,j} + \beta_2 Z_{2,j} + \beta_3 Z_{3,j} + \beta_4 Z_{4,j} + \epsilon_j \qquad (12)$$

where $X_j = \log I\left(\omega_{1,j},\omega_{1,j}\right)$, $\beta_0 = \log f_H(0,0), \beta_1 = -d_1$, $\beta_2 = -d_2, \beta_3 = -D_1, \beta_4 = -D_2$, $Z_{1,j} = \log\left|1-\mathrm{e}^{-i\omega_{1,j}}\right|^2$, $Z_{2,j} = \log\left|1-\mathrm{e}^{-i\omega_{2,j}}\right|^2$, $Z_{3,j} = \log\left|1-\mathrm{e}^{-is\omega_{1,j}}\right|^2$, $Z_{4,j} = \log\left|1-\mathrm{e}^{-is\omega_{2,j}}\right|^2$, and $\epsilon_j = \log\frac{I\left(\omega_{1,j},\omega_{1,j}\right)}{f_X(\omega_{1,j},\omega_{2,j})}$. The harmonic ordinates $\omega_{1,j}$ and $\omega_{2,j}$ are the jth element of ω_1 and ω_2 vectors, respectively, such that

$$
\begin{aligned}
\omega_1 &= \frac{2\pi}{N_1}(\underbrace{\underbrace{-m_1,\ldots,-1,1,\ldots,m_1},\ldots,\underbrace{-m_1,\ldots,-1,1,\ldots,m_1}}_{2m_2\text{times}}), \\
\omega_2 &= \frac{2\pi}{N_2}(\underbrace{-m_2,\ldots,-m_2}_{2m_1\text{times}},\underbrace{-m_2+1,\ldots,-m_2+1}_{2m_1\text{times}},\ldots,\underbrace{-1,\ldots,-1}_{2m_1\text{times}},\underbrace{1,\ldots,1}_{2m_1\text{times}},\ldots,\underbrace{m_2,\ldots,m_2}_{2m_1\text{times}});
\end{aligned}
$$

for $j = 1,2,\ldots,4m_1m_2$ where $m_1 = [\sqrt{N_1}]$ and $m_2 = [\sqrt{N_2}]$. We refer to Ghodsi and Shitan (2009) [Ghodsi and Shitan, 2009] more details. Now d_1, d_2, D_1 and D_2 may be estimates as $-\widehat{\beta_1}, -\widehat{\beta_2}, -\widehat{\beta_3}$, and $-\widehat{\beta_4}$ respectively by last squares estimation as in the usual multiple regression. The least squares optimality criterion minimizes the sum of squares of residuals between actual observed outputs and output values of the numerical model that are predicted from input observations.

2.2 Whittle Method

The Whittle method was studied by many authors to estimate the long memory parameters and it is based on evaluation of the Whittle likelihood function in temporal context. See for instance Kluppelberg and Mikosch [1993], Kluppelberg and Mikosch (1994) Claudia Kluppelberg [1994],Mikosch et al. [1995], Embrechts et al. [1997]. Assuming the process $\{X_{ij}\}_{i,j\in\mathbb{Z}_+}$ is a stationary Seasonal FISSAR model defined in (**??**), the Whittle likelihood function may expressed as

$$\sigma^2(\beta) = \frac{1}{N_1N_2}\sum_{j_1}\sum_{j_2}\frac{I\left(\omega_{1,j_1},\omega_{1,j_2}\right)}{g\left(\omega_{1,j_1},\omega_{1,j_2}\right)}, \qquad (13)$$

where $\beta = (d_1, d_1, D_1, D_2, \phi_{10}, \phi_{01}, \psi_{10}, \psi_{01})$ is the parameter vector of interest, $I\left(\omega_{1,j_1},\omega_{1,j_2}\right)$ is the periodogram define as

$$I\left(\lambda_1,\lambda_2\right) = \frac{1}{4\pi^2N_1N_2}\left|\sum_{k=1}^{N_1}\sum_{l=1}^{N_2}X_{k,l}e^{i(k\lambda_1+l\lambda_1)}\right|^2 \qquad (14)$$

and $g\left(\omega_{1,j_1},\omega_{1,j_2}\right) = \frac{4\pi^2}{\sigma_\varepsilon^2}f\left(\omega_{1,j_1},\omega_{1,j_2}\right)$ and $f(.,.)$ is the spectral density defined in (**??**).
Hence, an estimation procedure of β is to minimize (13) over β. The Whittle estimator is given by:

$$\hat{\beta} = \text{Argmin}\left\{\sigma^2(\beta)\right\} \qquad (15)$$

2.3 Exact Maximum Likelihood method

The maximum likelihood methods are considered among the most important estimation methods to estimate the long memory parameter. These are the most effective methods to estimate all parameters simultaneously. The advantages of the maximum likelihood estimation are discussed. Several researchers have used ML approach for estimating the statistical parameters in spatial models. On spatial linear models Mardia [1990] studied ML estimators for Direct Representation (DR), Conditional AutoRegression (CAR) models and Simultaneous AutoRegression (SAR) models in gaussian case. In addition, the author gives extension of the method in multivariate case, block data and missing value in lattice data. Earlier, Mardia and Marshall [1984] described the maximum likelihood method for fitting the linear model when residuals are correlated and when the covariance among the residuals is determined by a parametric model containing unknown parameters. Pardo-Igúzquiza [1998] used the maximum likelihood method for inferring the parameters of spatial covariances. The advantages of the ML estimation are discussed in Pardo-Igúzquiza [1998] for the multivariate distribution of the data and spatial analysis.
In this section, the maximum likelihood (ML) method for inferring the parameters of the Seasonal FISSAR model introduced Cisse et al. [2016] is examined. The causal moving average representation for the processes $\{X_{ij}\}$ in (1) and $\{W_{ij}\}$ in (2) are given in Cisse et al. [2016], Proposition (2.1):

$$X_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \phi_{10}^{k}\phi_{01}^{l}\psi_{10}^{m}\psi_{01}^{n} W_{i-k-ms_1,j-l-ns_2}, \tag{16}$$

where

$$W_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \phi_k(d_1)\phi_l(d_2)\phi_m(D_1)\phi_n(D_2)\varepsilon_{i-k-ms_1,j-l-ns_2}, \tag{17}$$

with

$$\phi_k(d_1) = \begin{cases} \dfrac{\Gamma(k+d_1)}{\Gamma(k+1)\Gamma(d_1)} & \text{if } k \in \mathbb{Z}_+ \\ 0 & \text{if } k \notin \mathbb{Z}_+ \end{cases} ; \qquad \phi_l(d_2) = \begin{cases} \dfrac{\Gamma(l+d_2)}{\Gamma(t+1)\Gamma(d_2)} & \text{if } l \in \mathbb{Z}_+ \\ 0 & \text{if } l \notin \mathbb{Z}_+ \end{cases} \tag{18}$$

and

$$\phi_m(D_1) = \begin{cases} \dfrac{\Gamma(m+D_1)}{\Gamma(m+1)\Gamma(D_1)} & \text{if } m \in \mathbb{Z}_+ \\ 0 & \text{if } m \notin \mathbb{Z}_+ \end{cases} ; \qquad \phi_n(D_2) = \begin{cases} \dfrac{\Gamma(n+D_2)}{\Gamma(n+1)\Gamma(D_2)} & \text{if } n \in \mathbb{Z}_+ \\ 0 & \text{if } n \notin \mathbb{Z}_+ \end{cases} \tag{19}$$

Our ML method is presented for both the two-dimensional seasonal fractionally integrated white noise process and the Seasonal FISSAR model. Asymptotic properties of maximum likelihood estimators for fractionally differenced AR model on a two-dimensional lattice were considered by Sethuraman and Basawa [1995], who showed that the usual asymptotic properties of consistency and asymptotic normality are satisfied under several conditions. Same procedures in Sethuraman and Basawa [1995] have been developed for obtaining maximum likelihood estimates of the parameters of the Seasonal FISSAR model and theirs asymptotic properties.

First we consider a $\{W_{ij}\}$ process defined as (2). We suppose that the stationary conditions are satisfied. Under this assumption and the normality assumption of the bi-dimensional white noise, construction of the likelihood function simply amounts to expressing the autocovariances of the process. Let $\{Y_{ij}\}$ define by

$$(1-B_1)^{d_1}(1-B_2)^{d_2} Y_{ij} = \varepsilon'_{ij} \tag{20}$$

The process $\{W_{ij}\}$ can be rewritten as

$$W_{ij} = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} \varphi_k(d_1)\varphi_l(d_2) Y_{i-s_1k,j-s_2l} \tag{21}$$

Therefore

$$\mathrm{Cov}\left(W_{ij}, W_{i_1j_1}\right) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{k_1=0}^{+\infty}\sum_{l_1=0}^{+\infty} \varphi_k(d_1)\varphi_l(d_2)\varphi_{k_1}(d_1)\varphi_{l_1}(d_2)\mathrm{Cov}\left(Y_{i-s_1k,j-s_2l}, Y_{i_1-s_1k_1,j_1-s_2l_1}\right) \tag{22}$$

The likelihood function of the sample $W = (W_{ij}, i = 1, ..., n; j = 1, ..., n)$ is seen to be

$$L_W = (2\pi)^{-n^2/2} |\Sigma_W|^{-1/2} \exp\left[-\frac{1}{2} W^T \Sigma_W^{-1} W\right],$$

where Σ_W is the variance-covariance matrix of X determined by (22).
The likelihood function of the Seasonal FISSAR $X = (X_{ij}, i = 1, ..., n; j = 1, ..., n)$ sample defined in a $n \times n$ regular lattice will be defined by the same way.

2.4 MLE method based on Whittle function

Here we consider the maximum likelihood method based on approximated Whittle function that usually gives a good estimator. This estimator is a parametric procedure due to Whittle with extension in the spatial context for the the regular lattice. The estimator is based on the periodogram and it involves the function

$$L(\beta) = \iint_{-\pi}^{\pi} \frac{I(\lambda_1, \lambda_2)}{f_X(\lambda_1, \lambda_2, \beta)} \, d\lambda_1 d\lambda_2, \tag{23}$$

where $f_X(\lambda_1, \lambda_2, \beta)$ is the known spectral density function at frequencies λ_1 and λ_2 and β denotes the vector of unknown parameters. The estimator is the value of β which minimizes the function $L()$. For the Seasonal FISSAR process the vector β contains the parameter d_1, d_2, D_1, D_2 and also all the unknown autoregressive parameters. For computational purposes, the estimator is obtained by using the discret form of $L()$:

$$L(\beta) = \frac{1}{4N_1N_2} \sum_{j_1} \sum_{j_2} \left\{ \log f_X(\omega_{1,j_1}, \omega_{1,j_2}) + \frac{I(\omega_{1,j_1}, \omega_{1,j_2})}{f_X(\omega_{1,j_1}, \omega_{1,j_2}, \beta)} \right\}, \tag{24}$$

We have that the maximum likelihood estimator of the long memory parameters in temporal case is strongly consistent, asymptotically normally distributed and asymptotically efficient in the Fisher sense (Dahlhaus,1989 Dahlhaus [1989] and Yajima,1985 Yajima [1985].
With the Monte Carlo study, we can show empirically the consistency and the asymptotic normality of the estimators.

3 Simulations and Discussions

To demonstrate the performance of the proposed methods several experiments was performed. The Monte Carlo study is designed to check variability of the Regression method in comparison with the classical Whittle method and Whittle MCMC procedures. The simulation results give the average values, the corresponding sample standard deviation (sd), the root mean square error (RMSE) and the mean absolute error (MAE) of the estimation procedures based on 500 replications. All calculations were carried out with fixed seasonal period $s = 4$ of the following models for various sample sizes summarized in Table (1).

	innovation parameters				memory parameters			
Parameters	ϕ_{10}	ϕ_{01}	ψ_{10}	ψ_{01}	d_1	d_2	D_1	D_2
Model 1	0.10	0.15	0.10	0.20	0.10	0.10	0.10	0.20
Model 2	0.10	0.15	0.10	0.20	0.10	−0.10	0.10	0.20
Model 3	0.10	0.15	0.10	0.20	0.10	−0.10	0.10	−0.20
Model 4	0.10	0.15	0.10	0.20	−0.10	−0.10	−0.10	−0.20

Table 1: Data generating processes

Which one to choose? This article provides several techniques for estimating parameters of the Seasonal FISSAR model. MLE is of fundamental importance in the theory of inference and is a basis of many inferential techniques in statistics. The first motivation behind ML estimation is to estimate all parameters simultaneously, unlike our proposal regression method and Wittle method. On the other hand referring to the computational times the Regression method bases on Log-periodogram would be preferred. unfortunately the AR parameters are typically not accurately estimated. However, the computational complexity of maximum likelihood (if there are $N \times N$ sampling points, then Σ is an $N \times N$ matrix, and the process can be slow if N is large) may be outweighed by its convenience as a very widely applicable method of estimation. Generally, it is suggested to try all proposal estimation approaches and compare.

4 Conclusion

The paper discuss and investigates the procedures estimation of the Seasonal FISSAR model. The classical Whittle and a log-regression method are used to estimate the memory parameters of the model. For estimating the innovation parameters and memory parameters simultaneously , the exact maximum likelihood method and the MLE based on Whittle function are considered.

Several directions of extending the Seasonal model and estimating methods are immediate. In this work we focused on a lattice setup, the essentials of this new model can be carried over to irregularly spaced data, in which case, weight matrices must be appropriately chosen for each location.

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