Linear and Non-linear Boundary Crossing Probabilities for Brownian Motion and Its Application in Predicting Bankruptcy

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Abstract

The prediction of financial distress and bankruptcy has been recently characterized as one of the more important problem facing both business and government. In this manuscript, a simple method based on boundary crossing probability of working capital process for predicting bankruptcy in median or small business is introduced. The method is based on finite Markov chain imbedding technique. A set of data from 183 manufacture companies of USA has been used to illustrate the method. The same data is also used to compare it against the more traditional methods: Altman's (1968) multivariate discriminant analysis and Ohlson's (1980) logistic regression. The numerical results show the proposed method performs very well.

1 Introduction

Bankruptcy in a company is an event that can produce substantial losses for owners, creditors, investors and workers. It can result in a chain reaction that affects other businesses and potentially becomes a social and economic problem for both the government and community. Financial bankruptcy prediction models can help to reduce such losses by providing the earliest possible warning to interested parties. Then action can be taken, such as cost cutting, or refinancing, or restructuring or even a merge with another company, in order to stave off bankruptcy.

According to Altman (1993), the first statistical study on business failure was done in 1935 by Smith and Winakor during the great depression and then by Meravin (1942). Beaver (1966) applied the uni-variate discriminant analysis (UVDA) using financial ratio to predict business failure. Altman (1968) extend Beaver's UVDA into multi-variate linear discriminant analysis (MLDA) using five financial ratios from company's financial statements for predicting business failure. The method was often referred as Z-score. It could be described as finding a linear discriminant function

$$Z = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 \tag{1}$$

with ratios x_1 = working capital/total assets, x_2 = retained earnings/total assets, x_3 = sales/total assets, x_4 = earning before interest and tax/total assets, and x_5 = market value of equity/total debt. These ratios are available from quarterly financial statement given by the company. In another word, the method is accounting based. The method has high accuracy in predicting at one year, and the decline in accuracy was significant. He also found *the working capital ratio to be the only significant predictor of business failure*. Ohlson (1980), Lo (1986) and Theodossiou (1993) proposed to use logistic regression for predicting business failure. Mathematically, the logistic regression method can be described as follows:

$$y = \log \frac{p}{1-p} = \sum_{j=1}^{N} \beta_j x_j \tag{2}$$

where p is the failure probability of a company and y is log of odds ratio and linear regression on covariates x_j, usually financial ratios. Lo (1986) and Theodossiou (1993) made claims that the logistic regression model with proper selecting the predictors performs at least as well as multivariate linear discriminant analysis.

Apart from above mentioned two popular statistical methodologies, it is also worth mentioning several others and recent development. For example Odom and Sharda (1990) proposed to use a neural network method to predict business failure. They showed that the neural network methodology is more robust than multivariate linear discriminant analysis. Bryant (1997) proposed a case-based seasoning approach to bankruptcy prediction. Mckee (2000) developed a bankruptcy prediction model via the rough set theory.

Recently Vassalou and Xing (2004) proposed a stochastic method studying the probability of default risk in equity returns. Hillegeist et al (2004) extended the method to assess the probability of bankruptcy. The method is based on the well-known Black-Scholes-Merton option pricing model. The probability of bankruptcy during the period $[0, T]$ is defined as the probability that the market value of assets, $V_A(t)$, is less than the face value of the liability X at time T; i.e.

$$P(V_A(T) < X), \tag{3}$$

where $V_A(t)$ is the market value at time $t \in [0, T]$ and $\log V_A(t)$ is normally distributed

$$\log V_A(t) \sim N(\log V_A(0) + (\mu - \delta - \frac{\delta^2}{2})t, \sigma_A^2(t)), \tag{4}$$

$V_A(0)$ is current market value of assets, μ stands for the expected return on assets, δ is dividend rate and $\sigma_A(t)$ is the volatility of asset returns. In order to enumerate the probability of bankruptcy, they used the European call option equation to estimate the parameters μ, δ and σ_A, and suggested the threshold X equals current liabilities plus 1/2 of long term liabilities which drew from Vassalou and Xing (2004). For application, this method has several drawbacks: (i) it is only applicable for companies which have options, (ii) the companies having options are usually large and stable companies, and (iii) the option prices are often biased in favorite of stock firm.

In this manuscript, we propose a stochastic method based on modeling the working capital as a Brownian motion with drift. The probability of bankruptcy is cast as boundary crossing probability (BCP) of working capital process $X(t)$ crossing the threshold X_0 in the time period $[0, T]$. Finite Markov chain imbedding technique by Fu and Lou (2003), and result by Fu and Wu (2010) are used to obtain the BCP. Numerical performance of proposed method has been carried out and compared with MLDA and LR to illustrate the theoretical result.

2 Stochastic Modeling Working Capital

As pointed out by Altman (1968), the working capital ratio is the only significant predictor among all the ratios for predicting business failure. We model the working capital $X(t)$ following a Brownian motion:

$$X(t) = \mu_t + \sigma_t W(t) \tag{5}$$

for the time period $[0, T]$, where μ_t is the trend (drift) of the working capital, σ_t is volatility of the working capital and $W(t)$ is standard Brownian motion. Under the above model, the probability of bankruptcy for a company in a time period $[0, T]$ is defined as boundary crossing probability that the working capital $X(t)$ is less than the threshold X_0; i.e.

$$P(X(t) < X_0, \text{ for some } t \in [0, T]). \tag{6}$$

It follows from eq. (5) the above boundary crossing probability equals to boundary crossing probability of standard Brownian motion with non-linear boundary

$$P(W(t) < \frac{X_0 - \mu_t}{\sigma_t}, \text{ for some } t \in [0, T]).$$

Given the parameters μ_t and σ_t and the threshold X_0, using the finite Markov chain imbedding technique and recent result of Fu and Wu (2010) the above probability of bankruptcy can be computed via the following theorem.

Theorem 2.1. *There exists a family of homogeneous Markov chains* $\{\hat{W}_n(t)\}$ *such that*

(i) $\hat{W}_n(t) \xrightarrow{\mathscr{D}} W(t)$*, as* $n \to \infty$*, for all* $t \in [0, T]$*,*

(ii) given two boundaries $a(t) < b(t)\, \forall t \in [0, T]$*, and* $a(0) < 0 < b(0)$*, then*

$$\begin{aligned} &P(W(t) \le a(t) \text{ or } W(t) \ge b(t), \text{ for some } t \in [0, T]) \\ &= 1 - \lim_{n \to \infty} \boldsymbol{\xi}_o \left(\prod_{i=1}^{n} \boldsymbol{A}_i \right) \mathbf{1}', \end{aligned} \tag{7}$$

where $\xrightarrow{\mathscr{D}}$ *stands for convergence in distribution,* $\boldsymbol{\xi}_0$ *is initial probability* $P(X(0) = c) = 1$*, and* $\boldsymbol{A}_i$ *are essential transition probability matrices of Markov chain* $\{\hat{W}_n(t)\}$ *with boundaries* $a(t)$ *and* $b(t)$ *as absorbing states, and*

(iii) $|P(a(t) < W(t) < b(t), \forall t \in [0, T]) - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} A_i\right) \mathbf{1}'| \leq \frac{C}{\sqrt{n}}$,

where C *is an unknown constant.*

We will neither show how to construct the family of Markov chains nor prove the results here. The construction and proof will be given in appendix. For the one-sided boundary crossing probability, we take $a(t) = -H$ and $H \to \infty$ (or $b(t) = H$ and $H \to \infty$) in our computation. This method will be referred as boundary crossing probability method (BCPM).

3 Numerical Comparison

In order to check the performance of proposed BCPM, the following numerical comparison with LR and MLDA is carried out using a set of US data which consists of 183 small manufacturing companies with revenues less than \$20 million. The working capital data was gathered for years 2001 to 2006 from quarterly financial statements.

For the MLDA and LR, five ratios were used for equations (1) and (2) to predicting the bankruptcy, respectively. For BCPM, we assume that the trend of working capital is linear $\mu_t = \alpha + \beta t$ and volatility is constant $\sigma_t = \sigma$

Table 1: Comparison of misclassification rates of the BCPM with $X_0 = -2TA$, MLDA and logistic regression methods with small-sized companies

Prediction (T=-2TA)	1-Year Prediction (2004)			3-Year Prediction (2006)		
Methods	BCPM	LR	MLDA	BCPM	LR	MLDA
Mis/Total ♯ of Non-Failed	0/93	2/93	26/93	2/73	4/73	26/73
% Misclassification Rate of Non-Failed	0%	2.15%	27.95%	2.74%	5.48%	35.61%
Mis/Total ♯ of Failed	5/14	11/14	5/14	23/34	28/34	14/34
% Misclassification Rate of Failed	35.71%	78.57%	35.71%	67.64%	82.35%	41.17%
Total Misclassification Rate	35.71%	80.72%	63.67%	70.39%	87.83%	76.79%

for the period $[0, T]$. Least squares estimators $\hat{\alpha}, \hat{\beta}$, and $\hat{\sigma}$ with threshold $X_0 = -2\times$total asset $(-2TA)$ were needed for computing the probability of bankruptcy using eq. (7) for each company. If the predicted probability of bankruptcy is greater than 0.5 we define the company failed and otherwise. The following Table 1 provides the performance of three methods for 1-year and 3-year predictions in terms of two types misclassification rates (type I and type II errors) and also the total misclassification rate.

Clearly, the BCPM performs much better than both MLDA and LR in both the 1-year and 3-year predictions. Particularly, the performance of BCPM is significant better in both Type I and Type II errors than other two methods. One may note that the misclassification rates of all three methods increase considerably from 1-year prediction to 3-year prediction. This implies the applications of any of those methods should not be extended beyond a 3-year period.

In view of simplicity in structure and better performance of BCPM, we would like to provide further technique remarks to lighten the method:

(i) Theoretically speaking, one should select the threshold X_0 which minimizes the sum of type I and type II errors, or linear combination of two types of errors for each company. Practically this is impossible due to the fact that no data is available for estimating the threshold. The threshold $X_0 = -2\times$total asset is selected empirically which minimizes the sum of observed type I and type II errors of the group of 183 companies.

(ii) It seems that the BCPM can be extended to involve two or more variables, for example working capital and cashflow, which may perform better than univariate BCPM. However selecting threshold and finding the boundary crossing probability for non-linear boundaries for two or high dimension Brownian motion remains a hard and challenging task.

(iii) One may note that the 183 manufacture companies in our study are relatively small business. Actually the method is expected to work for the median or large size companies. The group of companies in the study should be more or less homogeneous in structure, otherwise using the same threshold for all the companies may produce poor results.

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