Re-connecting probability and reasoning about data in secondary school teaching

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Introduction

In this paper, it will be argued that probability, despite its pluralistic epistemology, has been portrayed in curricula in a narrowly defined manner. Such a view of probability has become increasingly irrelevant when school-level exploration of data is exploiting the use of technology to draw inferences that apparently tell the story behind the data without reference to probability. It is necessary therefore to stress in curricula an alternative meaning for probability, one that is closer perhaps to how probability is used by statisticians in problem-solving. In this paper, probability as a modelling tool is explored through new technological developments that offer opportunities to re-connect probability to reasoning about data and problem-solving.

A pluralistic epistemology

Probability and statistics have, at least until recently, been clearly connected since statistical analysis attempts to draw conclusions about uncertain situations. Yet probability as an idea is difficult to pin down as perhaps evidenced by the late development of probability as a formally defined piece of mathematics the ongoing controversies about what it means (Hacking, 1975):

> ...we may readily confirm the fact that for all our advances in mathematical technology, a good many aspects of that dual concept of probability (frequentist and subjective) have been there from the beginning. The theories of today seem to compete in a space of possible theories that can be discerned even in the earliest years of our concept. (p. 16)

The many ways of thinking about probability represent differing epistemological roots and applications. Consider throwing a simple die. It is possible to identify a finite and countable sample space, and so, assuming independence and symmetry, a random variable might be defined classically by associating the outcomes [1, 2, 3, 4, 5, 6] with a probability of $\frac{1}{6}$ in each case and comparing the number of favourable cases to number of possible cases. Alternatively, the frequentist view might regard the probability of a 6 on a die as the limit of the proportion of outcomes that are 6's in an every increasing number of trials, observing experimentally that the proportion may be tending towards a limit of $\frac{1}{6}$. Third, probability might be seen as a rational measure of belief based on one's knowledge of the a priori probabilities and the conditions of the experiment so that previous experience or theoretical analysis might have led to a prior belief that the probability of a 6 on a die is $\frac{1}{6}$ but for a particular die this estimate may change in the light of further results. These strands in the development of the concept of probability can lead to different perspectives on some central ideas in this domain and, according to some researchers, a confusion amongst learners that has been labelled *epistemological anxiety* (Wilensky, 1997).

Probability in the curriculum

Perhaps such confusion cannot be entirely avoided in curricula; choices have to be made about which views of probability to stress and in what order. If we examine current curricula, it seems that the choice has been made to present probability as either theoretical (such as in the first example above, the classical approach) or experimental (as in the second example, the frequentist view).

At secondary level, the National Curriculum in the UK has focussed on (i) identifying finite countable sample spaces; (ii) relative frequency as an estimate of probability; (iii) the use of subjective estimations; (iv) the addition of probabilities for mutually exclusive events; (v) the multiplication of probabilities of independent events; (vi) the use of tabulation or tree diagrams for managing compound events. The examples in curriculum statements and textbooks are dominated by the use of coins, dice, spinners and drawing balls or sweets from a bag. This is not surprising since in such examples: a) the sample space is relatively easily identified, hence facilitating (i) above; b) experiments making few demands on equipment can be conducted in the classroom to demonstrate how the so-called *experimental probability* approximates the *theoretical probability* as in (ii) above; problems based around calculations that use the algebra of probabilities and representations such as tree diagrams are not difficult to invent to enable (iv), (v) and (vi). The reference to subjective probabilities in (iii) tends to be limited to encouraging students to associate 0 with the impossible, 1 with the certain and values between with varying degrees of uncertainty. Subjective probabilities are quickly left behind once this numerical scale has been established and are certainly not subjected to algebraic manipulation. The notion that an experimental probability (relative frequency) approximates the theoretical probability positions probability ontologically as a *thing* which exists, even if only in the imagination, and which can be glimpsed, if somewhat unsatisfactorily, through experiments.

This is not new. Probability is a relatively recent addition to the curriculum and although the extent to which probability has been seen as part of the primary school curriculum in the UK has varied over the last two decades (and is now almost non-existent again), there has been little variation in the position of probability in the secondary school curriculum. However, expectations around what we might expect from children when handling data are changing rapidly. The invention of new techniques in Exploratory Data Analysis (EDA) (Tukey, 1977) alongside developments in innovative software, such as Fathom (www.keypress.com/x5656.xml) and Tinkerplots (www.keypress.com/x5715.xml), have allowed researchers to explore how children can draw inferences from data without recourse to a classical understanding of probability. For example, Ben Zvi (2004) studied 13-year-old students in an experimental school as they sought to make comparisons between the lengths of English-American and Hebrew names. By studying in detail two students, Ben Zvi identified a number of developmental steps in how the students harnessed variability. These began with stages of where to focus attention, how to describe variability, conjecturing possible explanations for that variability and how to measure variability.

The term informal inferential reasoning (IIR) has been coined to describe the learning processes, whether as a precursor to learning classical inference or as an essential piece of equipment for the modern statistically literate citizen, who needs to be able to reason with and about data. For example, Zeiffler et al (2008) offer one helpful definition that describes IIR "as the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples" (p. 44). For example, a teaching approach first implemented by Bakker has been called *growing samples* (Bakker & Gravemeijer, 2004). A student might first collect data about themselves and close friends. The teacher might ask the student whether these data can be used to make predictions about the class. When the class data has been collected, a further challenge might ask whether predictions can be made about conclusions about the whole year of students, and so on, widening perhaps to the school and town etc. As the sample grows in tis approach so does the population but the focus remains throughout on whether inferences can be drawn about the population from the sample.

Such approaches demonstrate how the statistics curriculum is responding to the opportunities offered by EDA, new technology and an understanding of IIR. Yet the probability curriculum is not changing. As a result, while the teaching and learning of statistics takes on an enquiry-based problem-solving stance, where students act as data detectives, the pedagogy of probability is ever more isolated in its strange world of coins, spinners and dice as tools for demonstrating in a rough and ready way the *existence* of theoretical probability.

Yet as argued above, there is a plurality of epistemologies for probability and as a result there are opportunities available for curricula to develop around different choices and emphases about how probability is presented. Note that, compared to some other curricula, such as those in Australia and the US, there is less emphasis in the UK on modelling and the social uses of chance (see Table 20.2, p. 914 in Jones et al, 2007). In Australia for example, gambling is perceived as a widespread problem and some educators have developed units around the teaching of probability with this social problem in mind. Perhaps this example illustrates how a modelling approach could facilitate the re-connecting of probability to statistics in such a way that probability becomes interesting to school students in its relevance to social issues (see also later when probability in risk is discussed).

Probability as a modelling tool

Biehler (1994) has argued:

> A major point is that the ontological debate of whether something 'is' deterministic or not may not be useful, rather, a situation can be described with deterministic and with probabilistic models and one has to decide what will be more adequate for a certain purpose. (p. 4)

From this perspective, the behaviour of the coin, spinner or die can be seen as determined if *only* one knew enough about the many forces affecting their behaviour in which case the outcomes would be entirely predictable. In practice, this is an unlikely state of affairs and it is likely that one would be interested instead in adopting a probabilistic model. Hence, probability might be seen as a means for modelling the coins behaviour.

In the journal, Teaching Statistics Volume 32, there was an interesting debate, which might be further informed by taking a modelling perspective on probability. Ridgeway and Ridgway (2010) challenged children to make sense of sequences of outcomes of tosses from a coin, which was, according to the researchers, *fair*. When inspecting a particularly unlikely sequence, one child asserted that the coin was perhaps not fair. This response was marked as 'incorrect'. In his letter responding to the article, Goldstein (2010) argued *inter alia* that the researchers should not treat this response as wrong, claiming that he himself might well have given the same answer! The evidence provided might well lead to the inference that the coin was not fair in the same way that one might reject a null hypothesis. Ridgway and Ridgway presumably had in mind that even fair coins can generate freakish sequences of outcomes. Goldstein wanted to point out that coins should not be assumed to be fair since in practice throws may not be independent or indeed the probability at each throw may not be $\frac{1}{2}$. If probability were seen as a modelling tool, then the behaviour of the coin as witnessed might be modelled through a series of independent events with the probability of a head at each trial as $\frac{1}{2}$ and occasionally freakish sequences will be observed, or alternatively the behaviour might be modelled as a series of trials with some degree of dependency and/or non-equiprobability. The exercise of choice by the student over how to represent the behaviour positions probability quite differently and arguably offers to the student more creative activity.

Of course, if the modelling meaning of probability was stressed in the curriculum, it is debatable whether there is much advantage in maintaining the current emphasis on coins, spinners, dice and balls drawn from a bag. Perhaps, in days gone by when children played board games, there was some natural relevance in such contexts but, now that games take place in real time on screens, probability has much more relevance as a tool for modelling computer-based action and for simulating real-world events and phenomena. The claim here is that students would find much more purposeful building models with probability and that probability could take on new utility (Ainley et al, 2006). As a result, perhaps probability could gain a similar level of meaningfulness as do statistical constructs when students explore data in EDA. The next two sections of this paper offer examples of recent developments in software that position probability as a modelling tool and through which exploring data and modelling with probability can be connected.

Basketball

http://people.ioe.ac.uk/dave%5Fpratt/Dave_Pratt/Software.html

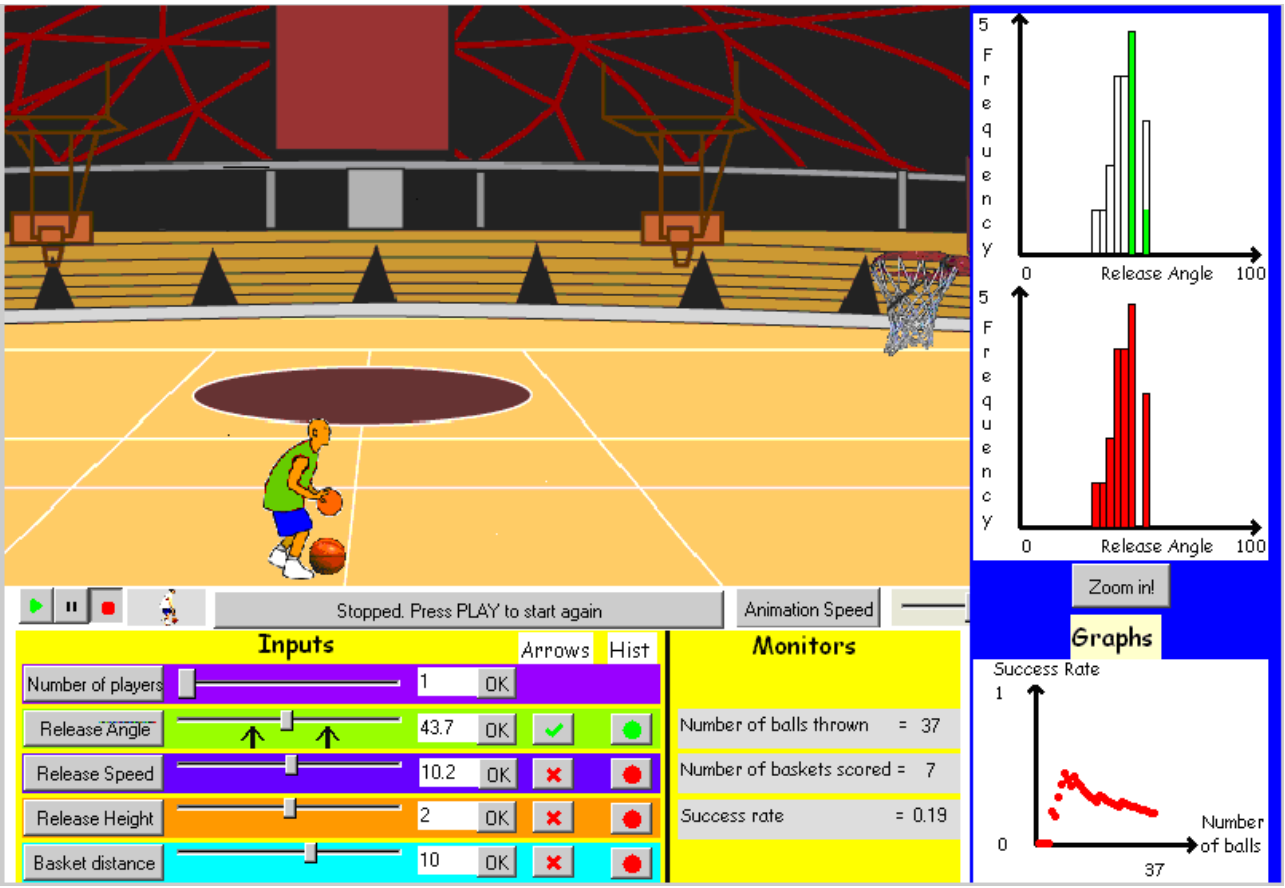

Figure 1: The player attempts to throw the ball into the basket by setting the sliders for release angle, speed, height and distance.

Basketball was developed as part of Theodosia Prodromou's doctoral thesis (2008). The notion was to explore two perspectives on distribution. It was anticipated that, because of the nature of the curriculum, secondary school students might articulate a *datacentric* perspective on distribution, as would be consistent with an EDA approach. An alternative view would be to recognise the probabilistic aspects of distribution. When statisticians posit variation in data as partially explained by main or higher order effects and unexplained variation as random error, they envisage a model of how the data are generated – out of various effects and random error. In a computer-based simulation, a probability distribution can take on the utility of generating data, akin to how the statistician envisages the model of the data. In her thesis, Prodromou built a computer-based simulation of a basketball player trying to throw the ball into the basket. Underlying the simulation was a mechanism for determining the trajectories of the balls. That mechanism could either be a determined system based on Newton's Laws of Motion or a stochastic model based around a probability distribution, which allowed variation in the trajectories. She set out to design these tools as a window (Noss and Hoyles, 1996) on: (i) the datacentric perspective, which envisages distribution as a set of data about the trajectories; (ii) the *modelling* perspective, which imagines the probability distribution as generating the trajectories. Crucially, Prodromou was interested in whether and how students might connect the two perspectives.

Through a design research approach (Cobb et al, 2003), Basketball was developed in such a way that students (age 14-16 years) could attempt to throw an on-screen ball into a basket by setting parameters, such as release angle, release speed, height of throw and distance from basket (see Figure 1). In Figure 1, the arrows have been set to ‘ON’ for ‘Release Angle’ but remain ‘OFF’ for the other parameters. In fact, at the start, all arrows would be set to ‘OFF’. Under these circumstances, the path of the ball would be determined precisely by the parameter values. Once a successful throw had been found, all subsequent throws would continue along the exact same trajectory unless the parameters were changed. However, once the arrows have been switched ‘ON’ (for ‘Release Angle’ in Figure 1), there would be some random variation in the trajectories of the ball. The size of that variation is set by the distance between the two arrows, which can also be dragged to different positions on the slider. On the right hand side of Figure 1, there are two histograms and one line graph. The lower histogram shows the frequency with which any particular release angle was chosen randomly by the computer. The higher histogram shows how often throws of different release angles were successful. The line graph traces the overall success rate as the number of throws increases. The histograms provide data to allow inferences to be drawn about the best throws on the basis of a datacentric perspective. The handle of the slider points to the mean average value in the probability distribution that generates particular values for the release angle. The arrows point to the spread of that probability distribution. (It is also possible to introduce skewness by positioning the arrows asymmetrically.)

In Prodromou and Pratt (2006), it is explained how the students harnessed ideas about causality to explain how the arrows ‘caused’ the balls to vary. In the thesis itself, Prodromou (2008) described two directions of connection between the datacentric and modelling perspectives. Some students regarded the probability distributing as emerging out of the data, a sort of target to which the data head. Others described the probability distribution as setting the intention before the data were generated. What is interesting about this work for the argument in this paper is that the Basketball simulation offers a way to think about how, by conceiving of probability as a modelling tool, there are opportunities within simulations for probability to have utility. In Prodromou’s work, the task is not to find the probability with which a successful basket might be made but to attempt to model realistic basketball throwing such that the player might be successful on some occasions but not always, and that the amount of error reflects the real performance of basketball players of various levels of skill.

Tinkerplots 2

www.keypress.com/x5715.xml

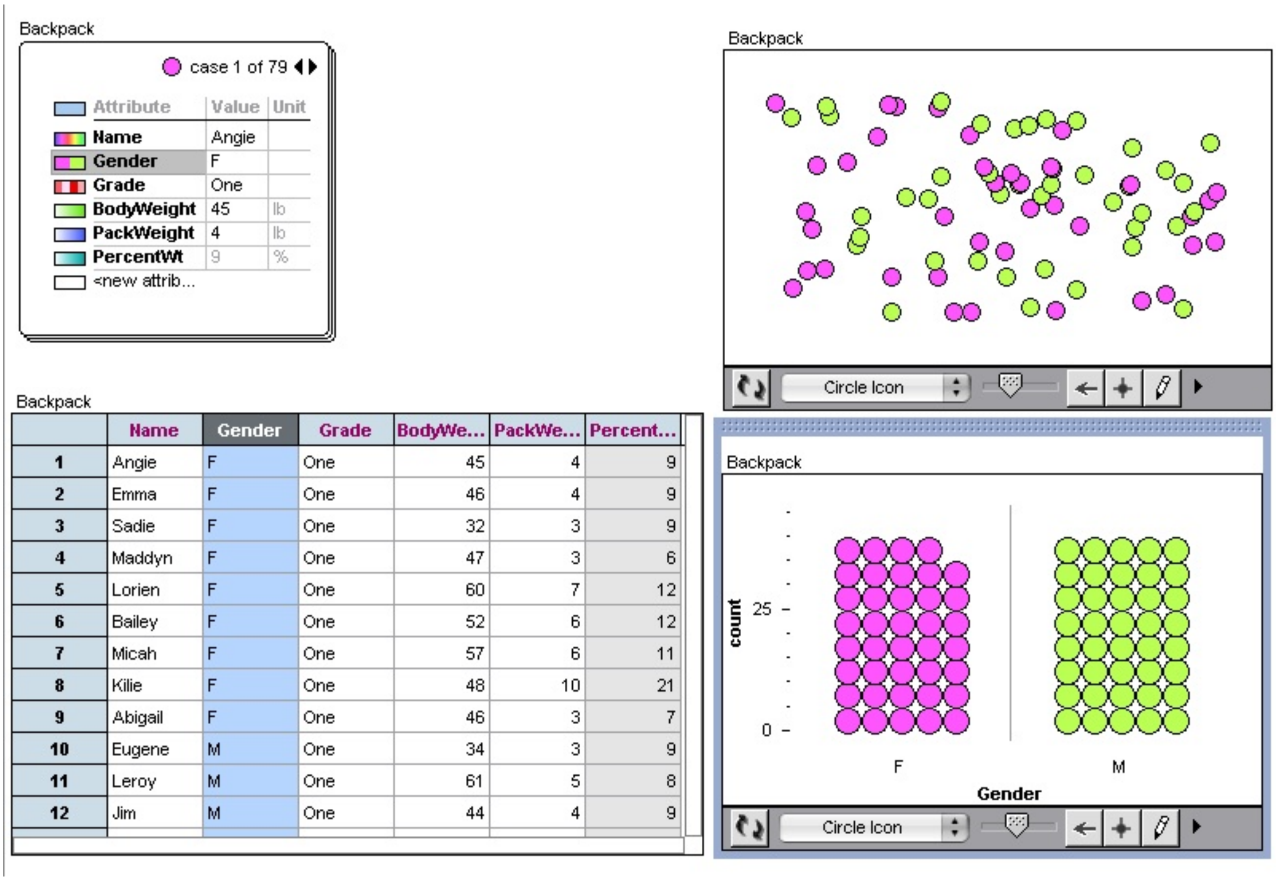

Figure 2: In Tinkerplots, the emphasis is on the case, which can be viewed as a card in a dataset (top left) or as a row in a table (bottom left) or as a graphical object (top right). In the bottom right, the cases have been organised by placing the attribute 'Gender' on the bottom axis and by stacking.

Tinkerplots has been developed by a team led by Cliff Konold. Konold's early interests were in probability where he developed a simulation tool called ProbSim. For many years, Konold's focus switched to the challenge of EDA. Tinkerplots offered an intuitive interface for young children to explore how to organise and compress data through graphs and numerical measures. The emphasis is on the specific case, which can be viewed as a card in a data set or as a row in a spreadsheet-like table or an arbitrary graphical object on the screen. The cases can be organised and re-organised dynamically by the child who makes choices about what attributes of the data to position on one or two axes and whether to stack the objects (see Figure 2).

More recently, Konold's team have re-focused on probability by offering new tools that can exploit probability as a modelling tool. Tinkerplots 2 provides so-called *samplers* (see Figure 3). The sampler is essentially a non-conventional representation of a probability distribution and comes in several different forms. One option is that the student can choose to sample using a mixer (top left in Figure 3). Here the student decides what balls to place in the mixer and then one is chosen at random. Amongst the other options are the possibilities to set up a sampler as a spinner by defining the sizes of the sectors (top right), a histogram by setting the heights of the bars (bottom left), or a probability density function by drawing a curve (bottom right). Once a sampler has been established, it can be run as many times as desired to generate data which can then be organised and represented using the tools described above.

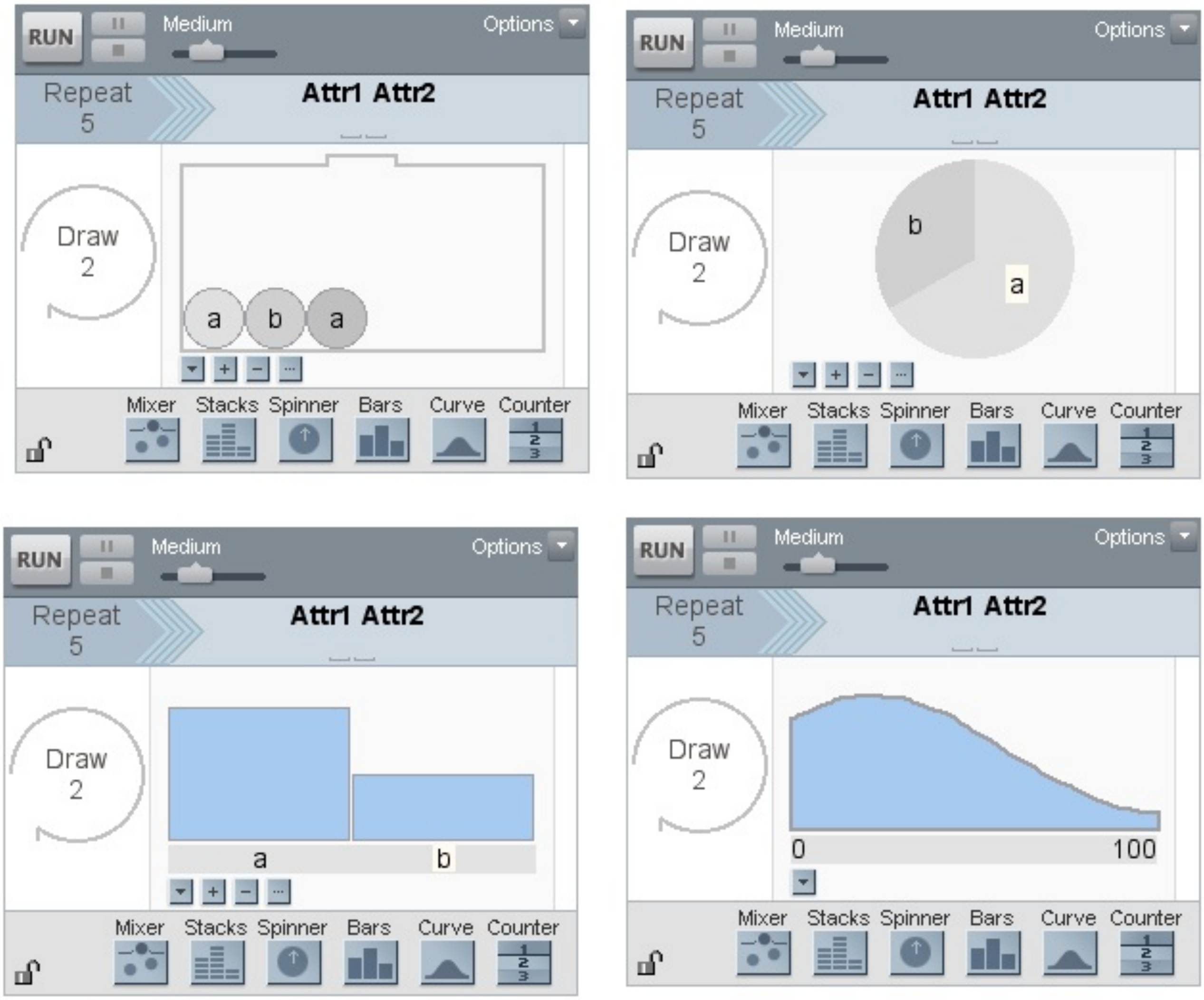

Figure 3: Four examples of samplers in Tinkerplots 2.

Early research using these tools is providing very interesting data. In Konold et al (2007), students age 12-14 years built ‘factories’ that generated people’s heights and measurement errors. They reported on a number of challenges such as how it was non-trivial for these students to conceive of objects as comprising a set of attributes. A case, such as a person, is a holistic entity and replacing this entity that can be perceived through all of our senses by a cluster of pieces of data was not a natural step, but an important one since it lies at the heart of modelling. Moving over to a modelling approach as advocated in this paper would bring along with it a set of new challenges and the study by Konold et al alerts curriculum designers to a fundamental one. Nevertheless, it is reasonable to argue that such difficulties need to be addressed in the modern world and are perhaps being hidden from view by current curricula approaches. In another study, Konold & Kazak (2008) provided middle-school students with a footprint supposedly left at a crime scene. Members of the class measured the length of the footprint and noted variation in their measurements, presumably due to error.

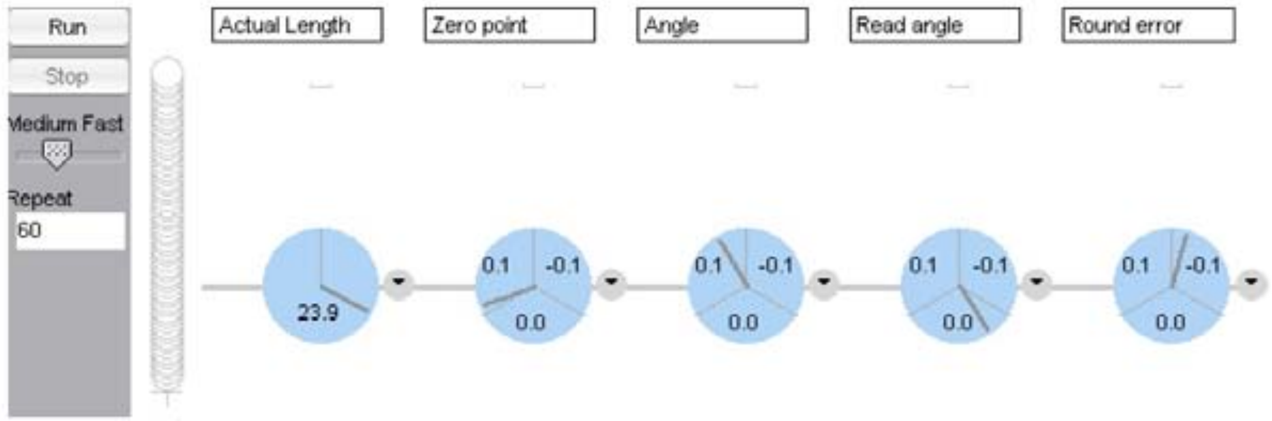

Figure 4: The mean average measurement is represented as a constant spinner and the measurements errors are represented as four independent spinners, which accumulate to give an overall error (Konold & Kazak, 2008, Figure 2, p.8).

They then used Tinkerplots 2 to model those errors using for example a set of four connected spinners, each of which contributed a value of $\pm 0.1 or 0$ to the most common measurement of 23.9 (see Figure 4). There was surprise that the distribution of data generated was not flat but peaked at 23.9. The students were able to explain after seeing the distribution generated that extreme measures were rather hard to obtain. Konold and Kazak discuss the role of signal and noise in this experiment. The mean value is one type of signal and the errors modelled by the spinners generate noise. However, when the model is run 60 times on three separate occasions, there is a stability in the peaked data distribution, which could also be regarded as a signal, and there is some variation in the detail of the shape, which could be regarded as noise. What we learn from this report is that a modelling approach to probability places distribution in the foreground but not as a fixed pre-determined truth but as an entity open to debate, a choice made by the modeller to create reasonable approximations of phenomena, where the nature of the approximation lies in the relationship between signal and noise.

Conclusion

The argument presented in this paper starts from the position that probability, more than any other school-level mathematical construct, offer a range of different epistemologies, which can be seen as a source of confusion for students but also as a set of choices for curriculum designers. In practice, currently the probability curriculum arguably reflects concerns and issues that were more relevant to earlier generations where the focus was almost entirely on classical inference and where coins, spinner and dice were clearly relevant to children. The position of probability in the school curriculum is being exacerbated by developments out of EDA in the statistics curriculum where there is a contrast between: (i) novel enquiry-based approaches where data are mined to search for trends and patterns, and (ii) the probability curriculum where probability is presented as a fixed starting point from which various calculations can be made.

A possible response would be for the probability curriculum to develop around the notion of probability as a modelling tool that could be used to build models in computer-based simulations, akin to the video-games that engage children and adolescents of today. Two examples of tools from recent research are offered. The Basketball simulation demonstrated how students might make connections between a datacentric and a modelling perspective on distribution by relating sliders that represent parameters of the probability distribution to the shape and position of histograms of data. Research with Tinkerplots 2 is beginning to collect examples of students using probability as a tool to describe variation in real or imagined phenomena. In so doing, some of the challenges in moving to a modelling-based probability curriculum are being identified. For example, students do not easily reduce cases to sets of attributes.

Figure 5: The possible consequences of Deborah having the operation are modelled by setting probabilities for the overall success-rate and for possible side-effects. When the model is run many times, it is possible to eyeball Deborah's futures using the colour coded chart of possible outcomes.

A modelling approach to probability would fit more comfortably with the use of statistics in disciplines other than mathematics and may enable statistics to be understood by students as a cross-curricular subject as well as a key idea in mathematics. Recent research on risk offers another avenue for probability as a modelling tool. Pratt et al (in press) have been researching mathematics and science teachers' subject and pedagogical knowledge about risk, which has recently become a significant part of the science curriculum and an application of probability in the mathematics curriculum in the UK. They have developed a tool, called *Deborah's Dilemma* (www.riskatioe.org/), in which the user is expected to example a complex scenario in which a young woman, Deborah, suffers from a back condition, which an operation might cure but which might result in minor or major consequences (such as paralysis or death). The user is able to model what might happen to Deborah by drawing on rich information about Deborah's condition including sometimes-conflicting information from a range of different doctors and surgeons. The probabilities that the user might insert into the model are far from fixed. The values used will inevitably be compromises based on the information available and the consequences of making different choices can be explored in the simulator.

Presenting probability in the curriculum as a modelling tool will inevitably bring with it certain new challenges in how children learn but these difficulties can be embraced as essential steps to overcome in the development of students who will engage fully in modern society. By connecting probability to statistics and to simulated and real phenomena, such a change in the curriculum reflects the use of mathematics and statistics in contemporary society and promises an enquiry-based pedagogy, which students are more likely to find purposeful and relevant.

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ABSTRACT

The teaching of probability and of statistics in secondary schools in the UK, as in most countries, takes place rightly or wrongly in mathematics classes. As the mathematics curriculum becomes increasingly influenced by software developments that facilitate a focus on drawing inferences by manipulating and representing data as in Exploratory Data Analysis, probability retreats into an isolated world of coins, dice and spinners. Secondary school students are not being encouraged to see the wider relevance of probabilistic thinking. Very recent software developments promise the re-connection of probability and reasoning about data by offering probability as a modelling tool. Opportunities are now available to review how these developments might redirect the curriculum including the wider and earlier use of probability as a subjective measure of chance. This paper will review some of these possibilities.