The Optimal Design of Choice Experiments that Incorporate Ties

Bush, Stephen (1st author)
University of Technology, Sydney, School of Mathematical Sciences
PO Box 123
Broadway NSW (2007), Australia
E-mail: Stephen.Bush@uts.edu.au

Introduction

Stated choice experiments are a common method for modelling decision making behaviour. These experiments are used in many disciplines, including marketing, health economics, tourism, and public policy. Louviere et al. (2000) and Train (2003) provide a good introduction to the area.

In a stated choice experiment each respondent is presented with a series of choice sets, the same series for each respondent, and for each choice set they are asked to choose one of the options presented. In general we will describe the items to be compared by k attributes, where we assume that the q^{th} attribute has ℓ_q levels, represented by $0, 1, \ldots, \ell_q - 1$, and that each choice set in an experiment has m options. We assume that no choice set contains a repeated option and that no choice set is repeated in an experiment. We also assume that the options are generic and unlabelled. Provided that the choice experiment has been correctly designed, these responses can be used to estimate the effects of each of the attributes on the probability that an item is selected, and to estimate the effects of the interactions of any two of the attributes on the probability that an item is selected.

In these circumstances, if respondents are forced to choose one of the items in each choice set, a number of results about the optimal design exist; see Street and Burgess (2007) for a summary of this and related work. This work assumed the multinomial logit (MNL) model. In choice sets of size 2 the MNL model coincides with the Bradley–Terry model.

Sometimes respondents do not have a 'best' option, but find that two, or more, items are equally attractive. Davidson (1970) extended the Bradley–Terry model so that ties could be accommodated in a paired comparison experiment. Bush et al. (2010) extend this model to allow for a fixed, but arbitrary, number of options within a choice set. The authors referred to this extended model as the generalised Davidson ties model.

In this paper, we perform simulations to compare the ability of different designs to efficiently estimate model parameters for the generalised Davidson ties model. In the next section, we introduce the model, and state the optimality result proved in Bush et al. (2010). We then perform a simulation study to compare D–optimal designs under the null hypothesis of equal selection probabilities to an alternative design, for various values for the model parameters.

The Generalised Davidson Ties Model

Suppose that we present m items $\{T_{i_1}, \ldots, T_{i_m}\} = C$ to the respondent. Then we can estimate a merit, π_i, for each of these items. If we use the MNL model, the probability that the item $T_i \in C$ is chosen is $P(T_i|C) = \pi_i / (\sum_{a=1}^{m} \pi_{i_a})$. In this set up, the respondent is forced to choose a single item from the choice set, even if they actually find two or more items equally attractive. In this section we discuss the generalisation of the MNL model to accommodate ties introduced in Bush et al. (2010).

If we allow ties then, when $m > 2$, the respondent is not only permitted to find pairs of items in the choice set equally attractive, but is also permitted to state that larger subsets of the items in the choice set are equally attractive. Bush et al. (2010) uses Davidson's argument that the merit of finding a set of items equally attractive is proportional to the geometric mean of the item merits, and also assumes that the proportionality is constant across choice sets and is strictly positive. We denote this proportionality constant by ν. If this constant is equal to zero then this means that no respondent has stated that any of the items in any of the choice sets are equally attractive, and in this case the MNL model should be used instead.

Bush et al. (2010) lets the merit of item T_i be π_i, the merit of the set of items T_{i_1} and T_{i_2} be $\nu\sqrt{\pi_{i_1}\pi_{i_2}}$. Let the merit of the set of items T_{i_1}, T_{i_2} and T_{i_3} be $\nu\sqrt[3]{\pi_{i_1}\pi_{i_2}\pi_{i_3}}$ and so on until the merit of the respondent finding all of the m items in the choice set equally attractive being $\nu\sqrt[m]{\pi_{i_1}\pi_{i_2}\dots\pi_{i_m}}$. Then, for a choice set $C=\{T_{i_1},T_{i_2},\dots,T_{i_m}\}$, the sum of the merits for each possible decision is D_C, where

$$D_C = \sum_{a=1}^{m}\pi_{i_a}+\sum_{x=2}^{m}\sum_{\{T_{j_1},\dots,T_{j_x}\}\subseteq C}\nu\sqrt[x]{\pi_{j_1}\dots\pi_{j_x}}.$$

We can then express the probabilities for each decision as

$$P(T_{i_1}|C)=\frac{\pi_{i_1}}{D_C},\quad P(\{T_{i_1},T_{i_2}\}|C)=\frac{\nu\sqrt{\pi_{i_1}\pi_{i_2}}}{D_C},\dots,\quad P(\{T_{i_1},T_{i_2},\dots,T_{i_m}\}|C)=\frac{\nu\sqrt[m]{\pi_{i_1}\pi_{i_2}\dots\pi_{i_m}}}{D_C}.$$

As in Street and Burgess (2007), we are interested in estimating the contrasts $B_\gamma\boldsymbol{\gamma}$ as well as ν. These contrasts, whose coefficients are in B_γ, may correspond to the main effects of the attributes, or two–factor interactions between attributes, or perhaps subsets of these. The results in Bush et al. (2010), as with those in Street and Burgess (2007), find designs that are optimal under the null hypothesis of equal selection probabilities, that is $\boldsymbol{\pi}=\mathbf{1}$. Bush et al. (2010) show that the information matrix for the estimation of $B_\gamma\boldsymbol{\gamma}$ and ν, where B_γ contains contrasts of the attributes, is

$$C(\boldsymbol{\pi}_0,\nu) = \begin{bmatrix} Q(m,\nu)\times C(\boldsymbol{\pi}_0)_{\text{MNL}} & \mathbf{0} \\ \mathbf{0} & \Lambda_{\nu\nu}(m,\nu)\end{bmatrix},$$

where

$$Q(m,\nu)=\frac{m+\nu\sum_{x=2}^{m}\left(\frac{m-x}{x(m-1)}\binom{m}{x}\right)}{m+\nu\sum_{x=2}^{m}\binom{m}{x}},$$

$C(\boldsymbol{\pi}_0)_{\text{MNL}}$ is the Fisher information matrix for the estimation of the MNL model using the same design, and $\Lambda_{\nu\nu}(m,\nu)$ is a 1×1 element which is independent of the design for a fixed value of m. Bush et al. (2010) use this relationship to prove the following theorem.

Theorem 1. (Bush et al. (2010)) For a set of p contrasts of the entries in $\boldsymbol{\gamma}$ and a constant but unknown ν, the D–optimal design for the estimation of the contrasts of $\boldsymbol{\gamma}$ over a set of competing designs $\mathfrak{X}$ when the MNL model is used will also be D–optimal for the estimation of the same contrasts and ν over the same set of competing designs when estimating the generalised Davidson ties model.

This theorem allows us to use the results that exist for the MNL model, such as those in Burgess and Street (2003) and Burgess and Street (2005) to find optimal designs under the null hypothesis when the generalised Davidson ties model is used.

Simulations of the generalised Davidson ties model

In this section we consider the performance of the generalised Davidson ties model under various model assumptions by carrying out a number of simulation studies. We assume that there are two attributes, both with two levels, and $m=3$ throughout. We consider two sets of values for the parameters. In the first set we assume that both main effects parameters, τ_1 and τ_2, are equal to 0 and the ties parameter $\nu=0.5$, and in the second set we assume that $\tau_1=1$ and $\tau_2=-1$ although $\nu=0.5$ still.

We find the locally optimal design for each set of values and compare the performance of each design with both sets of parameter values. The design in Table 1(a) is optimal for the estimation of the main effects of the attributes plus the ties parameter when $\tau_1 = \tau_2 = 0$, and $\nu = 0.5$, by Theorem 1 and Theorem 1 of Burgess and Street (2003). By an exhaustive search of the $2^4 - 1 = 15$ possible designs, the design in Table 1(b) is optimal for the estimation of the main effects of the attributes plus the ties parameter when $\tau_1 = 1$, $\tau_2 = -1$, and $\nu = 0.5$.

We first assume that $\tau_1 = \tau_2 = 0$, and $\nu = 0.5$ and compare the simulated distributions of the parameter estimates when the designs in Tables 1(a) and (b) are used in turn. Each simulation is modelled using the simulated responses from 150 respondents, and the summary statistics describe the distributions of the estimates from 1000 such simulations. The summary statistics for the first set of parameter values are provided in Table 2. We see that the distributions of the parameter estimates are symmetrically distributed. As expected, the variance of the parameter estimates for the design in Table 1(a) is smaller than that of the design in Table 1(b), even after taking into account the different numbers of choice sets in the two designs, illustrating the efficiency of the former design.

We now consider the performance of these two designs when $\tau_1 = 1, \tau_2 = -1$, and $\nu = 0.5$. Summary statistics for the distributions of the parameter estimates when the designs in Tables 1(a) and (b) are used are provided in Table 3. We see that, for both designs, the distributions of the parameter estimates seems to be unbiased and close to symmetric. The difference between the variances arising from the two designs is smaller in this case than when $\tau_1 = \tau_2 = 0$.

We now compare the ability of four different designs to estimate the main effects plus the two–factor interaction of the attributes and ν. The first two designs are those in Tables 1(a) and (b). The third design is shown in Table 4(a), and is optimal for the estimation of the main effects plus the two–factor interaction of the attributes and ν when $\tau_1 = \tau_2 = \tau_{12} = 0$, and $\nu = 0.5$, by Theorem 1 and Theorem 2 of Burgess and Street (2003). The final design, shown in Table 4(b), is locally optimal for the estimation of the main effects plus two–factor interaction of the attributes and ν when $\tau_1 = 1, \tau_2 = -1, \tau_{12} = -0.25$, and $\nu = 0.5$, by an exhaustive search.

We first consider the case where there is no significant interaction effect. We let $\tau_1 = 1$, $\tau_2 = -1$, $\tau_{12} = 0$, and $\nu = 0.5$. Summary statistics for the parameter estimates for the designs in Tables 1(a) and (b), and Table 4(a) and (b) are provided in Table 5.

The design in Table 4(a) gives parameter estimates with the smallest variance, and are also unbiased and symmetrically distributed. The designs in Tables 1(a) and 4(b) also give unbiased and symmetric parameter estimates, but with a larger variance than those from the design in Table 4(a), even after taking into account the different numbers of choice sets in the two designs. The design in Table 1(b) gives parameter estimates that are slightly biased towards 0, skewed, and with the largest variance of the four designs.

Now we consider the case where there is a significant interaction effect. Suppose that $\tau_1 = 1$, $\tau_2 = -1$, $\tau_{12} = -0.25$, and $\nu = 0.5$. Summary statistics for the parameter estimates when each of the four designs are used are provided in Table 6.

Again we see that the design in Table 4(a) gives parameter estimates with the smallest variance, and are also unbiased and symmetrically distributed. The designs in Tables 1(a) and 4(b) once again give unbiased and symmetric parameter estimates, but with a larger variance than those form the design in Table 4(a). The design in Table 1(b) once again gives parameter estimates that are slightly biased towards 0, skewed, and with the largest variance of the four designs.

In summary, we see that the designs that are optimal for the estimation of a certain set of effects appear to give parameter estimates that are unbiased and reasonably symmetric when estimating the same set of effects. When designs are used for the estimation of a different set of parameters to those that the design is constructed for, there may be bias in the parameter estimates.

References

Burgess, L.B., and Street, D.J., 2003. Optimal designs for 2^k choice experiments. Communications in Statistics – Theory and Methods 32(11), 2185–2206.

Burgess, L.B., and Street, D.J., 2005. Optimal designs for choice experiments with asymmetric attributes. Journal of Statistical Planning and Inference 134(1), 288–301.

Bush, S.A., and Burgess, L.B., and Street, D.J., 2005. Optimal designs for Stated Choice Experiments that Incorporate Ties. Journal of Statistical Planning and Inference 140(7), 1712–1718.

Davidson, R.R., 1970. On Extending the Bradley–Terry Model to Accommodate Ties in Paired Comparison Experiments. Journal of the American Statistical Association 65(319), 317–328.

Louviere, J.J. Hensher, D.A. and Swait J.D., 2000. Stated Choice Methods. Cambridge University Press.

Street, D.J., and Burgess, L., 2007. The Construction of Optimal Stated Choice Experiments. Wiley.

Train, K.E., 2003. Discrete Choice Methods with Simulation. Cambridge University Press.

Option 1	**Option 2**	**Option 3**
0 0	0 1	1 0
0 1	0 0	1 1
1 0	1 1	0 0
1 1	1 0	0 1

(a)

Option 1	**Option 2**	**Option 3**
0 0	1 0	1 1
0 1	1 0	1 1

(b)

Table 1: Optimal design for estimating main effects and ν when $\tau_1 = \tau_2 = 0$, and $\nu = 0.5$ (a) and when $\tau_1 = 1, \tau_2 = -1$, and $\nu = 0.5$ (b).

Parameter	**Simulated Mean (Standard Error)**	**Theoretical Variance**	**Simulated Variance**	**Simulated Skewness (Standard Error)**
Design in Table 1(a)				
τ_1	−0.00088(0.00165)	0.00278	0.00271	0.02920(0.07734)
τ_2	−0.00244(0.00165)	0.00278	0.00273	0.13122(0.07734)
Design in Table 1(b)				
τ_1	−0.00445(0.00237)	0.00556	0.00564	−0.04829(0.07734)
τ_2	−0.00303(0.00232)	0.00556	0.00536	−0.10379(0.07734)

Table 2: Summary statistics for $\tau_1 = \tau_2 = 0$, and $\nu = 0.5$.

Parameter	**Simulated Mean (Standard Error)**	**Theoretical Variance**	**Simulated Variance**	**Simulated Skewness (Standard Error)**
Design in Table 1(a)				
τ_1	1.00248(0.00216)	0.00320	0.00465	0.20895(0.07734)
τ_2	−0.99894(0.00226)	0.00320	0.00510	−0.21943(0.07734)
Design in Table 1(b)				
τ_1	1.01053(0.00417)	0.00545	0.01736	0.36056(0.07734)
τ_2	−0.99980(0.00307)	0.00544	0.00941	−0.18175(0.07734)

Table 3: Summary statistics for $\tau_1 = 1$, $\tau_2 = -1$, and $\nu = 0.5$.

Option 1	**Option 2**	**Option 3**
0 0	0 1	1 0
0 1	0 0	1 1
1 0	1 1	0 0
1 1	1 0	0 1
0 0	1 0	1 1
0 1	1 1	1 0
1 0	0 0	0 1
1 1	0 1	0 0
0 0	0 1	1 1
0 1	0 0	1 0
1 0	1 1	0 1
1 1	1 0	0 0

(a)

Option 1	**Option 2**	**Option 3**
0 0	0 1	1 0
0 0	0 1	1 1
0 0	1 1	1 0
1 1	1 0	0 1

(b)

Table 4: Optimal design for estimating main effects, two–factor interactions and ν when $\tau_1 = \tau_2 = \tau_{12} = 0$, and $\nu = 0.5$ (a) and when $\tau_1 = 1, \tau_2 = -1, \tau_{12} = -0.25$, and $\nu = 0.5$ (b).

Parameter	**Simulated Mean (Standard Error)**	**Theoretical Variance**	**Simulated Variance**	**Simulated Skewness (Standard Error)**
Design in Table 1 (a)				
τ_1	0.98950(0.00240)	0.00320	0.00574	0.14233(0.07734)
τ_2	−0.99129(0.00240)	0.00322	0.00578	−0.18005(0.07734)
τ_{12}	−0.02435(0.00233)	0.00321	0.00543	0.17092(0.07734)
Design in Table 1 (b)				
τ_1	0.99104(0.00482)	0.00549	0.02319	0.60368(0.07734)
τ_2	−0.96591(0.00446)	0.00729	0.01985	−0.28153(0.07734)
τ_{12}	−0.03756(0.00438)	0.00729	0.01919	0.30541(0.07734)
Design in Table 4 (a)				
τ_1	0.98549(0.00141)	0.00107	0.00198	0.08501(0.07734)
τ_2	−0.98395(0.00134)	0.00107	0.00180	−0.02499(0.07734)
τ_{12}	−0.02720(0.00133)	0.00107	0.00176	0.05257(0.07734)
Design in Table 4 (b)				
τ_1	0.98751(0.00282)	0.00400	0.00793	0.18660(0.07734)
τ_2	−0.98777(0.00272)	0.00411	0.00738	−0.10613(0.07734)
τ_{12}	−0.03117(0.00257)	0.00404	0.00661	0.17683(0.07734)

Table 5: Summary statistics for $\tau_1 = 1, \tau_2 = -1, \tau_{12} = 0$, and $\nu = 0.5$.

Parameter	**Simulated Mean (Standard Error)**	**Theoretical Variance**	**Simulated Variance**	**Simulated Skewness (Standard Error)**
Design in Table 1 (a)				
τ_1	0.99699(0.00236)	0.00329	0.00559	0.10950(0.07734)
τ_2	−0.99738(0.00239)	0.00331	0.00574	−0.22312(0.07734)
τ_{12}	−0.27001(0.00233)	0.00330	0.00541	0.10207(0.07734)
Design in Table 1 (b)				
τ_1	1.01073(0.00513)	0.00575	0.02628	0.57665(0.07734)
τ_2	−0.98728(0.00462)	0.00764	0.02132	−0.31215(0.07734)
τ_{12}	−0.28134(0.00442)	0.00764	0.01950	0.29241(0.07734)
Design in Table 4 (a)				
τ_1	0.99521(0.00137)	0.00110	0.00189	0.13158(0.07734)
τ_2	−0.99530(0.00138)	0.00110	0.00191	−0.07656(0.07734)
τ_{12}	−0.27166(0.00133)	0.00110	0.00176	0.08612(0.07734)
Design in Table 4 (b)				
τ_1	1.00344(0.00302)	0.00418	0.00912	0.23760(0.07734)
τ_2	−0.99695(0.00260)	0.00431	0.00677	−0.11871(0.07734)
τ_{12}	−0.27676(0.00253)	0.00422	0.00640	0.00381(0.07734)

Table 6: Summary statistics for $\tau_1 = 1, \tau_2 = -1, \tau_{12} = -0.25$, and $\nu = 0.5$.