On Asymptotic Higher-Order Expansions by a Two-Stage Procedure

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1. Introduction

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables from a certain distribution. In many sequential estimation problems of parameters of distributions, the expression of the so called "optimal" fixed sample size turns out to be

$$n_0 = \frac{q\theta}{h} \tag{1}$$

where q and h are known positive numbers, but θ is the unknown and positive nuisance parameter. Assume that $\theta > \theta_L$ where $\theta_L (> 0)$ is known to the experimenter.

As an example, suppose that the mean μ and the variance σ^2 of a distribution are both finite and unknown. Having recorded $X_1, \ldots, X_n$, one may estimate μ by $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ under the loss function $L_n = (\overline{X}_n - \mu)^2$. Then, the risk is given by $R_n = E(L_n) = \sigma^2/n$. For any preassigned $w > 0$, we hope that $R_n = \sigma^2/n \le w$, which is equivalent to

$$n \ge \sigma^2/w.$$

Hence, the optimal fixed sample size n_0 becomes σ^2/w, which corresponds to $h = w$, $\theta = \sigma^2$ and $q = 1$ in (1). Unfortunately σ^2 is unknown, so we can not use the optimal fixed sample size n_0. Thus, we use a sequential procedure. For this bounded risk problem, the asymptotic analyses when $w \to 0$ correspond to those as $h \to 0$ in (1).

In sequential estimation of the normal mean, Mukhopadhyay and Duggan (1997) showed second-order properties of the Stein-type two-stage procedure under the assumption that the unknown variance has a known and positive lower bound. The results were extended to a fairly general setup by Mukhopadhyay and Duggan (1999). In this paper, we consider the general two-stage procedure of Mukhopadhyay and Duggan (1999) and show its asymptotic higher-order properties. It will be seen that our higher-order approximations are more accurate than the second-order approximations of Mukhopadhyay and Duggan (1999). Our main theorems are described in Section 2. As an example, our results are applied to the above bounded risk estimation of the normal mean in Section 3.

2. Asymptotic theory

We consider the following two-stage procedure which is the one of Mukhopadhyay and Duggan (1999) with $\tau = 1$. Taking account of (1), the initial sample size is defined by

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$$(2) \qquad m \equiv m(h) = \max\left\{m_0,\ \left[\frac{q\theta_L}{h}\right]^* + 1\right\}$$

where m_0 is a preassigned positive integer and $[x]^*$ denotes the largest integer less than x. Based on the pilot sample $X_1, \ldots, X_m$, we consider an estimator $U(m)$ of θ satisfying $P\{U(m) > 0\} = 1$ and $E\{U(m)\} = \theta$. Further, suppose that

$$Y_m = \frac{p_m U(m)}{\theta} \text{ is distributed as } \chi^2_{p_m} \quad \text{with } p_m = c_1 m + c_2$$

where p_m is a positive integer with positive integer c_1 and integer c_2, and $\chi^2_{p_m}$ stands for a chi-square distribution with p_m degrees of freedom. Then,

$$m \to \infty \quad \text{and} \quad U(m) \xrightarrow{P} \theta \quad \text{as } h \to 0$$

where "$\xrightarrow{P}$" stands for convergence in probability. Let q^*_m be positive where

$$q^*_m = q + c_3 m^{-1} + O(m^{-2}) \quad \text{as } h \to 0$$

with some real number c_3. It follows from (1) and (2) that

$$\frac{m}{n_0} = \frac{\theta_L}{\theta} + O(n_0^{-1}) \quad \text{as } h \to 0.$$

From the pilot sample $X_1, \cdots, X_m$, we calculate $U(m)$ and define

$$(3) \qquad N \equiv N(h) = \max\left\{m,\ \left[\frac{q^*_m U(m)}{h}\right]^* + 1\right\}.$$

If $N > m$, then one takes the second sample $X_{m+1}, \ldots, X_N$. The total observations are $X_1, \ldots, X_N$. For the general two-stage procedure defined by (2) and (3), Mukhopadhyay and Duggan (1999) showed the following second-order efficiency property: as $h \to 0$, namely as $n_0 \to \infty$

$$(4) \qquad \psi + o(n_0^{-1/2}) \le E(N) - n_0 \le \psi + 1 + o(n_0^{-1/2}), \quad \text{where } \psi = \frac{c_3\theta}{q\theta_L}.$$

We shall give a higher-order efficiency property of the above two-stage procedure.

THEOREM 1. *We have as $h \to 0$*

$$E(N) - n_0 = \psi + \frac{1}{2} + O(n_0^{-1}), \quad \textit{where } \psi \textit{ is as in (4).}$$

REMARK 1. The relation (4) consists of inequalities, but our Theorem 1 consists of a equality. Therefore, our approximation is more accurate and gives a more explicit relation between the average sample number $E(N)$ and the lower bound θ_L through ψ than that of Mukhopadhyay and Duggan (1999). Further, our order term $O(n_0^{-1})$ in Theorem 1 is sharper than the term $o(n_0^{-1/2})$ in (4).

Throughout the remainder of this paper, we use the following notations:

$$\tilde{T} = \frac{q^*_m U(m)}{h} \quad \text{and} \quad S = [\tilde{T}]^* + 1 - \tilde{T}.$$

Then, (3) becomes $N = \max\{m, [\tilde{T}]^* + 1\}$. Suppose that g: $R^+ \to R^+$ is a three-times differentiable function and the third derivative $g^{(3)}(x)$ is continuous at $x = 1$. By Taylor's theorem, we have

$$\begin{aligned} g(N/n_0) &= g(1) + g'(1)n_0^{-1}(N - n_0) + (1/2)g''(1)n_0^{-2}(N - n_0)^2 \\ &\quad + (1/6)g^{(3)}(W)n_0^{-3}(N - n_0)^3 \end{aligned}$$

where W is a random variable such that $|W-1| < |(N/n_0)-1|$. Then we obtain the following theorem.

THEOREM 2. *If $\{g^{(3)}(W)n_0^{-3/2}(N-n_0)^3\,;\,0<h<h_0\}$ is uniformly integrable for some sufficiently small $h_0>0$, then as $h\to 0$*

$$E\{g(N/n_0)\} = g(1) + B_0 n_0^{-1} + O(n_0^{-3/2})$$

and

$$E\{g(N/n_0)\} = g(1) + B_0 n_0^{-1} + A_h n_0^{-3/2} + o(n_0^{-3/2})$$

where

$$B_0 = g'(1)\left(\psi + \frac{1}{2}\right) + g''(1)\frac{\theta}{c_1\theta_L}\,, \qquad A_h = g''(1)n_0^{-1/2}E\{(\tilde{T}-n_0)S\}$$

$$\textit{and}\quad |A_h| \le |g''(1)|\sqrt{\frac{\theta}{6c_1\theta_L}} + O(n_0^{-1/2})\,.$$

REMARK 2. Mukhopadhyay and Duggan (1999) showed that as $h\to 0$

$$g(1) + B_1 n_0^{-1} + o(n_0^{-1}) \le E\{g(N/n_0)\} \le g(1) + B_2 n_0^{-1} + o(n_0^{-1}),$$

where B_1 and B_2 are constants, depending on θ_L. Since we have $B_1 \le B_0 \le B_2$, our Theorem 2 gives a more accurate approximation than that of Mukhopadhyay and Duggan (1999).

3. Bounded risk estimation

In this section, we consider a sequence of i.i.d. random variables $X_1, X_2, X_3, \ldots$ from a normal population $N(\mu,\sigma^2)$ where $\mu\in(-\infty,\infty)$ and $\sigma^2\in(0,\infty)$ are both unknown. We assume that there exists a known and positive lower bound σ_L^2 for σ^2 such that $\sigma^2 > \sigma_L^2$. Having recorded $X_1,\ldots,X_n$, we define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \quad\text{and}\quad U(n) = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2 \quad \text{for}\ \ n\ge 2.$$

As stated in Section 1, we want to estimate μ by $\overline{X}_n$ under the loss function $L_n = (\overline{X}_n-\mu)^2$, where the risk is given by $R_n = E(L_n) = \sigma^2/n$. For any preassigned $w>0$, we hope that $R_n = \sigma^2/n \le w$, which is equivalent to

$$n \ge \frac{\sigma^2}{w} \equiv n_0.$$

Since we can not use the optimal fixed sample size n_0, we define a two-stage procedure. Let

$$m = m(w) = \max\left\{m_0,\ \left[\frac{\sigma_L^2}{w}\right]^* + 1\right\}, \tag{5}$$

where $m_0 \ge 4$ is a preassigned integer. By using the pilot observations $X_1,\ldots,X_m$, we calculate $U(m)$ and

$$N = N(w) = \max\left\{m,\ \left[\frac{b_m U(m)}{w}\right]^* + 1\right\}, \tag{6}$$

where $b_m = (m-1)/(m-3)$. It is easy to see that $P(N<\infty)=1$ for all μ, σ^2 and w. Once sampling stops at stage N, the risk is given by $R_N = E(\overline{X}_N-\mu)^2$. It follows from section 7c.6 of Rao (1973) that $R_N = E(\sigma^2/N) \leq w$ for all fixed μ, σ^2 and w. On the notations in Sections 1 and 2, note that $h = w$, $\theta = \sigma^2$, $q = 1$, $p_m = m - 1$ $(c_1 = 1,\ c_2 = -1)$ and $q_m^* = b_m = 1 + 2m^{-1} + O(m^{-2})$ with $c_3 = 2$. The following proposition follows immediately from Theorem 1.

PROPOSITION 1. *For the two-stage procedure defined by (5) and (6), we have as* $w \to 0$

$$E(N - n_0) = \psi + \frac{1}{2} + O(n_0^{-1}), \quad \textit{where } \psi = 2\sigma^2/\sigma_L^2 \,.$$

We can give an asymptotic higher-order expansion of the risk $R_N = E(\overline{X}_N - \mu)^2$ by using Theorem 1 and Theorem 2.

PROPOSITION 2. *Let* ψ *be as in Proposition 1. For the two-stage procedure defined by (5) and (6), we have as* $w \to 0$

$$R_N = w\left\{1 - \frac{1}{2}n_0^{-1} + A_w n_0^{-3/2} + o(n_0^{-3/2})\right\}$$

where

$$A_w = 2n_0^{-1/2} E\{(\tilde{T} - n_0)S\} \quad \textit{with} \quad |A_w| \leq 2\sqrt{\frac{\sigma^2}{6\sigma_L^2}} + O(n_0^{-1/2})\,,$$

$$\tilde{T} = \frac{b_m U(m)}{w} \quad \textit{and} \quad S = [\tilde{T}]^* + 1 - \tilde{T}.$$

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