Optimal Stein Rule in Spherical Models

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Abstract: In this approach, we construct the necessary and sufficient conditions in order a James-Stein type estimator outperforms the minimax estimator of the mean. Particularly we consider a class of spherically symmetric distributions and derive the dominating conditions under the quartic loss function. Multivariate Student's t and Slash distributions as two examples are also considered for checking the efficiency of the proposed model and specifying theoretical requirements.

Key words and phrases: James-Stein estimator, Quartic loss function, Spherically symmetric.

1 Introduction

Definition 1.1. *An absolutely continuous random n-vector $\boldsymbol{X} = (X_1, \cdots, X_n)'$ has elliptically contoured distribution with parameters $\boldsymbol{\theta} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathcal{S}(p)$ and $\psi(.) \in [0, \infty) \to \mathbb{R}$ if its characteristic function is $\phi(\boldsymbol{t}) = e^{i\boldsymbol{t}'\boldsymbol{\theta}}\psi(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$. Furthermore, if $\boldsymbol{X}$ possess a density, then it can be presented by*

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = d_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}} g\left[(\boldsymbol{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})\right], \tag{1}$$

where d_p is the normalizing constant, satisfying the condition $d_p = \int_0^\infty t^{\frac{p}{2}-1} g(t)dt < \infty$, where $g(.)$ is a non-negative Lebesgue measurable function. In this case we use the notation $\boldsymbol{X} \sim EC_p(\boldsymbol{\theta}, \boldsymbol{\Sigma}, g)$.

ECD provides highly impressive list of heavier/lighter tail alternatives to the multivariate Gaussian models. Some recent materials involving vector-variate distributional properties and inferential problem will be found entirely in the work of Muirhead (1982), Fang et al. (1990) and Fang and Zhang (1990). Some of the well known members of the multivariate spherically/elliptically contoured family of distributions are the multivariate normal, Kotz Type, Pearson Type VII, Multivariate Student's t (MT), Multivariate (MS), Logistic, Multivariate Bassel distributions.

Lemma 1.1. *(Chu, 1973) If $\boldsymbol{X} \sim EC_p(\boldsymbol{\theta}, \boldsymbol{\Sigma}, g)$, then*

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_0^\infty W(t) N_p\left(\boldsymbol{\theta}, t^{-1}\boldsymbol{\Sigma}\right) dt = \int_0^\infty W(t) \frac{t^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}} \exp\left[-\frac{t}{2}(\boldsymbol{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})\right] dt,$$

where $W(t) = (2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}} t^{-\frac{p}{2}}\mathcal{L}^{-1}[f(s)]$, $\mathcal{L}^{-1}[f(s)]$ denotes the inverse Laplace transform of $f(s)$ with $s = (\boldsymbol{x}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})/2$.

The elliptical distributions are the parametric forms of the spherical symmetric distributions, which are invariant under orthogonal transformations and have equal density on sphere if densities exist. In this paper we actually deal with spherical models i.e. $\boldsymbol{\Sigma} = \boldsymbol{I}_p$.

Let $\boldsymbol{X} = (X_1, ..., X_p)$ distributed according to the model (1) when $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Following Fourdrinier et al. (2008) we basically engage with the problem of estimating $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)$ under the quartic loss function

$$L(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X})) = \sum_{i=1}^p (\delta_i(\boldsymbol{X}) - \theta_i)^4, \tag{2}$$

Where $\boldsymbol{\delta}(\boldsymbol{X}) = (\delta_1(\boldsymbol{X}), ..., \delta_p(\boldsymbol{X}))$ estimates $\boldsymbol{\theta} = (\theta_1, ..., \theta_p)$, and investigate the conditions for which an estimator $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{g}(\boldsymbol{X})$ dominates $\boldsymbol{X}$ for $p \geq 3$. We give an extension to the earlier work for the class of scale mixture of multivariate normal distributions. More important we show the robustness of the superiority conditions for the classes under study.

As it is noted in the earlier work, the quartic loss is neither quadratic nor spherically symmetric. Hence our results represent an interesting example of the stein effect whereby reasonably explicit dominating estimators can be obtained in a setting that is somewhat unusual.

To be honest we must say that all fundamental computations are followed by the utilities given in Fourdrinier et al. (2008).

2 Minimax estimators under quartic loss

The measurement associated with the quartic loss given by (2) is $R(\boldsymbol{\theta}, \boldsymbol{\delta}) = E_\theta[L(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X}))]$, Where E_θ denotes the expectation with respect to the sampling distribution (1). It is easy to show that for the minimax estimator $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$, $R(\boldsymbol{\theta}, \boldsymbol{\delta}^0(\boldsymbol{X})) = 3p$, since $\int W(t)dt = 1$. Following Stein (1981) we take the class of shrinkage estimators $\boldsymbol{\delta}(\boldsymbol{X})$ of $\boldsymbol{\theta}$ of the form $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{g}(\boldsymbol{X})$ into consideration where $\boldsymbol{g}$ is a function from $\mathbb{R}^p$ into $\mathbb{R}^p$. Note that from Stein identity and using Lemma 1.1 for the weakly differentiable function $\boldsymbol{g}$ we can immediately conclude that

$$\begin{aligned} E_{\boldsymbol{\theta}}\left[(X_i-\theta_i)g_i(\boldsymbol{X})\right] &= E_{\boldsymbol{\theta}}\left\{E_t\left[(X_i-\theta_i)g_i(\boldsymbol{X})|t\right]\right\} = \int_0^\infty E_{\boldsymbol{\theta}}\left[(X_i-\theta_i)g_i(\boldsymbol{X})\Big|t\right]W(t)dt \\ (3) \qquad &= \int_0^\infty t^{-1}\, E_{\boldsymbol{\theta}}\left[\frac{\partial}{\partial X_i}g_i(\boldsymbol{X})\Big|t\right]W(t)dt = \kappa^{(1,1,1)} \end{aligned}$$

where

$$(4) \qquad \kappa^{(l,j,k)} = \int_0^\infty \left(\frac{1}{t}\right)^l E_{\boldsymbol{\theta}}\left[\frac{\partial^j}{\partial X_i^j}g_i^k(\boldsymbol{X})\Big|V=t\right]W(t)dt.$$

More precisely

$$(5) \qquad \kappa^{(i)} = \int_0^\infty \left(\frac{1}{t}\right)^i W(t)dt.$$

Lemma 2.1. *Assume that $\boldsymbol{g}$ is a three times weakly differentiable function from $\mathbb{R}^p$ into $\mathbb{R}^p$ satisfying $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right] < \infty$, for every $1 \le i \le p$. Then, under quartic loss (2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is*

$$\eth g(\boldsymbol{X}) = \sum_{i=1}^p \left[g_i^4(\boldsymbol{X}) + 6\kappa^{(1)}g_i^2(\boldsymbol{X}) + 12\kappa^{(2,1,1)} + 4\kappa^{(1,1,3)} + 6\kappa^{(2,2,2)} + 4\kappa^{(3,3,1)}\right].$$

Proof: First consider that

$$\begin{aligned} \Delta_\theta &= R(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X})) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) = E_\theta\left[L(\boldsymbol{\theta}, \boldsymbol{\delta}(\boldsymbol{X})) - L(\boldsymbol{\theta}, \boldsymbol{\delta}^0)\right] \\ &= E_\theta\left\{\sum_{i=1}^p \left[g_i^4(\boldsymbol{X}) + (4(X_i-\theta_i)g_i^3(\boldsymbol{X}) + 6(X_i-\theta_i)^2 g_i^2(\boldsymbol{X}) + 4(X_i-\theta_i)^3 g_i(\boldsymbol{X}))\right]\right\}. \end{aligned}$$

Applying extended Stein's identity (3) repeatedly, we find

$$\begin{aligned} E_\theta\left[(X_i-\theta_i)g_i^3(\boldsymbol{X})\right] &= \kappa^{(1,1,3)}, \quad E_\theta\left[(X_i-\theta_i)^2 g_i^2(\boldsymbol{X})\right] = \kappa^{(1,1,2)} + \kappa^{(2,2,2)}, \\ E_\theta\left[(X_i-\theta_i)^3 g_i(\boldsymbol{X})\right] &= 3\kappa^{(2,1,1)} + \kappa^{(3,3,1)}. \end{aligned}$$

Gathering all the above terms, the result follows. □

From lemma 2.1 one can find that any estimator $\boldsymbol{\delta}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{g}(\boldsymbol{X})$ dominates $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ under the quartic loss (2) as soon as $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right] < \infty$, for every $1 \le i \le p$, is satisfied and

$$(6) \qquad \eth g(\boldsymbol{x}) \le 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^p$$

with strict inequality on a set of positive Lebesgue measure.

3 Class of James-Stein Estimators

Two typical classes of Shrinkage Stein-type estimators include well-known classes of James-Stein estimators. We basically involve with two types of shrinkage functions. Item one includes the one in (6) resulting on the following James-Stein (JS) estimator

$$\boldsymbol{\delta}_a^{JS}(\boldsymbol{X}) = \boldsymbol{X} - \frac{a}{\|\boldsymbol{X}\|^2}\boldsymbol{X}. \tag{7}$$

For better understanding of the finiteness condition $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right] < \infty$, we turn our attention to the shrinkage function $\boldsymbol{g}(\boldsymbol{X}) = \boldsymbol{g}_a(\boldsymbol{X}) = -\left(\frac{a}{\|\boldsymbol{X}\|^2}\right)\boldsymbol{X},\ a > 0$. For the specified class in (7), applying Theorem 1 of Bock et al. (1983) we have

$$E_{\boldsymbol{\theta}}\left(\frac{g_i^4(\boldsymbol{x})}{a^4}\right) \le E_{\boldsymbol{\theta}}\left(\frac{1}{\|\boldsymbol{x}\|^4}\right) = E_t E_{\boldsymbol{\theta}}\left(\frac{1}{\|\boldsymbol{x}\|^4}\bigg| t\right) = \int_0^\infty t^2 E_{\boldsymbol{\theta}}\left[\chi_p^{-2}\left(t\|\boldsymbol{\theta}\|^2\right)\bigg| t\right]^2 W(t)dt < \infty \tag{8}$$

for $p > 4$ and such distribution in which $\kappa^{(-2)} < \infty$.

Item two deals with the shrinkage function $\boldsymbol{g}_{a,b}(\boldsymbol{X}) = -\left(\frac{a}{\|\boldsymbol{X}\|^2+b}\right)\boldsymbol{X}$ (with $a > 0$ and $b > 0$) resulting on the following JS estimator

$$\boldsymbol{\delta}_{a,b}^{JS}(\boldsymbol{X}) = \boldsymbol{X} - \frac{a}{\|\boldsymbol{X}\|^2 + b}\boldsymbol{X}. \tag{9}$$

Now consider the JS class of estimators defined by (8). For the finiteness condition $E_{\boldsymbol{\theta}}\left[g_i^4(\boldsymbol{X})\right] < \infty$ consider that

$$\begin{aligned} E_{\boldsymbol{\theta}}\left(\frac{g_i^4(\boldsymbol{x})}{a^4}\right) &\le E_{\boldsymbol{\theta}}\left(\frac{\|\boldsymbol{x}\|^4}{b^4}\right) = \frac{1}{b^4}\int_0^\infty t^{-2} E_{\boldsymbol{\theta}}\left[\chi_p^2\left(t\|\boldsymbol{\theta}\|^2\right)\bigg| t\right]^2 W(t)dt \\ &= \frac{1}{b^4}\int_0^\infty t^{-2}\left[p(p+2) + 2(p+2)t\|\boldsymbol{\theta}\|^2 + t^2\|\boldsymbol{\theta}\|^4\right] W(t)dt \\ &= \frac{1}{b^4}\left[p(p+2)\kappa^{(2)} + 2(p+2)\|\boldsymbol{\theta}\|^2\kappa^{(1)} + \|\boldsymbol{\theta}\|^4\right] \\ &< \infty, \end{aligned} \tag{10}$$

provided that $\kappa^{(i)} < \infty$, $i = 1, 2$ and $\|\boldsymbol{\theta}\|^2 < \infty$.

Lemma 3.1. *Assume that $p \ge 5$. Under quartic loss (2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}_a^{JS}(\boldsymbol{X})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is expressed as*

$$\begin{aligned} \eth\boldsymbol{g}_a(\boldsymbol{X}) &= a\Bigg[\left(a^3 + 24\kappa^{(1)}a^2 + 144\kappa^{(2)}a + 192\kappa^{(3)}\right)\frac{\sum_{i=1}^p X_i^4}{\|\boldsymbol{X}\|^8} + 6\kappa^{(1)}\left(a - 2t(p-2)\right)\frac{1}{\|\boldsymbol{X}\|^2} \\ &\quad -12\left(a^2\kappa^{(1)} - (p-10)a\kappa^{(2)} - 2\kappa^{(3)}(p-8)\right)\frac{1}{\|\boldsymbol{X}\|^4}\Bigg]. \end{aligned}$$

Proof: The result follows from Lemma 2.2 of Fourdrinier et al. (2008) and Lemma 2.1. □

Theorem 3.1. *Under the model (1), the JS estimator given by (7) dominates $\boldsymbol{\delta}^0(\boldsymbol{X})$ under quartic loss (2), for all $\boldsymbol{\theta} \in \mathbb{R}^p$, provided $p \ge 7$, and*

$$0 < a < min\left\{2\kappa^{(1)}(p-2), sup\{s \ge 0/h(s) = 0\}\right\},$$

where $h(s) = s^3 + 12\kappa^{(1)}s^2 + 18sp\kappa^{(2)} - 12\kappa^{(3)}(p - 4 - 2\sqrt{2})(p - 4 + 2\sqrt{2})$.

Proof: Under conditioning, the risk difference Δ_θ is bounded above by

$$\begin{aligned} & aE_t\Big\{E_{\boldsymbol\theta}\Big[(a^3+24t^{-1}a^2+144at^{-2}+192t^{-3})\frac{1}{\|\boldsymbol X\|^4}+6t^{-1}(a-2t^{-1}(p-2))\frac{1}{\|\boldsymbol X\|^2} \\ & -12(a^2t^{-1}-(p-10)at^{-2}-2t^{-3}(p-8))\frac{1}{\|\boldsymbol x\|^4}\Big]\Big\} \\ = \; & aE_t\Big\{E_{\boldsymbol\theta}\Big[(a^3+12t^{-1}\,a^2+12a\,t^{-2}(p+2)+24pt^{-3})\frac{1}{\|\boldsymbol X\|^4} \\ & +6\,t^{-1}(a-2t^{-1}(p-2))\frac{1}{\|\boldsymbol X\|^2}\Big]\Big\}. \end{aligned} \tag{11}$$

Thus applying Lemma A.3 of Fourdrinier et al. (2008), by the assumption $a < 2\kappa^{(1)}(p-2)$, the bound in (11) reduces to

$$a\left[(a^3+12\kappa^{(1)}a^2+18ap\kappa^{(2)}-12\kappa^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2}))\right]E_{\boldsymbol\theta}\left[\frac{1}{\|\boldsymbol X\|^4}\right].$$

setting $h(a)=a^3+12\kappa^{(1)}a^2+18ap\kappa^{(2)}-12\kappa^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2})$, it is clear that $h(0) = -12\kappa^{(3)}(p-4-2\sqrt{2})(p-4+2\sqrt{2})<0$ since $p\geq 7$ and $\kappa^{(1)}>0$. Furthermore this cubic polynomial is increasing in a and hence negative on the interval $[0,a_0]$ where a_0 is its smallest root $a_0>0$. Finally, for $0<a<min\{2\kappa^{(1)}(p-2),a_0\}$, we have $\Delta_{\boldsymbol\theta}<0$, which is the desired domination result. □

Theorem 3.2. *Under the model* (1)*, the JS type estimator in* (7) *dominates* $\boldsymbol\delta^0(\boldsymbol X)$ *under quartic loss* (2)*, for all* $\boldsymbol\theta\in\mathbb{R}^p$*, provided* $p\geq 5$ *and* $0<a\leq 2\kappa^{(1)}(p-4)$*.*

Proof: Applying Lemma A.2 of Fourdrinier et al. (2008) and continuing in the same way as Theorem 3.1, the result follows. □

Theorem 3.3. *Under the model* (1)*, the risk* $R(0,\boldsymbol\delta_a^{JS})$ *of the JS estimator at* $\boldsymbol\theta=\mathbf 0$ *under quartic loss* (2) *is*

$$\frac{3}{(p+2)}\left(\kappa^{(2)}p\,(p+2)-4ap\,\kappa^{(1)}+6a^2-4\frac{a^3}{\kappa^{(1)}(p-2)}+\frac{a^4}{\kappa^{(2)}(p-2)(p-4)}\right)$$

and it is finite provided $p-4>0$*.*

Proof: Similar to the computations in the proof of Proposition 2.2 of Fourdrinier et al. (2008), the risk of JS estimators at $\mathbf 0$ under quartic loss (2) is given by

$$\begin{aligned} R(\mathbf 0,\boldsymbol\delta_a^{JS}) &= E_t\left\{E_{\boldsymbol\theta}\left[\sum_{i=1}^p\left(1-\frac{a}{\|\boldsymbol X\|^2}\right)^4X_i^4\right]\right\} \\ &= pE_t\left\{E_{\boldsymbol\theta}\left[\frac{X_i^4}{\|\boldsymbol X\|^4}\right]E_{\boldsymbol\theta}\left[\|\boldsymbol X\|^4\left(1-\frac{a}{\|\boldsymbol X\|^2}\right)^4\right]\right\}, \end{aligned}$$

since $Y_i|t=\left.\frac{X_i^2}{\|\boldsymbol X\|^2}\right|t$ is independent of $\|\boldsymbol X\|^2|t$ for $i=1,...,p$. As the distribution of $Y_i|t$ is *Beta* $(1/2,(p-1)/2)$ and the distribution of $\|\boldsymbol X\|^2|t$ is $t^{-1}\chi_p^2$,

$$E_\theta\left[\frac{X_i^4}{\|\boldsymbol X\|^4}\right] = E_tE_{\boldsymbol\theta}[Y_i^2]=E_t\left[\frac{B(\frac12+2,(p-1)/2)}{B(\frac12,(p-1)/2)}\right]=\frac{3}{p\,(p+2)},$$

since $\int_0^\infty W(t)dt=1$ and

$$\begin{aligned} E_{\boldsymbol\theta}\left[\|\boldsymbol X\|^4\left(1-\frac{a}{\|\boldsymbol X\|^2}\right)^4\right] &= E_tE_{\boldsymbol\theta}\left[\|\boldsymbol X\|^4-4a\|\boldsymbol X\|^2+6a^2-4\frac{a^3}{\|\boldsymbol X\|^2}+\frac{a^4}{\|\boldsymbol X\|^4}\right] \\ &= \int_0^\infty\Big[t^{-2}\,p\,(p+2)-4\,a\,p\,t^{-1}+6a^2-4\frac{a^3}{t^{-1}(p-2)} \\ &\quad +\frac{a^4}{t^{-2}(p-2)(p-4)}\Big]W(t)dt. \end{aligned}$$

Simplifying the above result, completes the proof. □

Similar to the Lemma 3.1, we have the following parallel result to the Lemma 3.1 of Fourdrinier et al. (2008) under the model (1).

Lemma 3.2. *Assume that $p \geq 3$. Under quartic loss (2), an unbiased estimator of the risk difference Δ_θ between $\boldsymbol{\delta}^{JS}_{a,b}(\boldsymbol{X})$ and $\boldsymbol{\delta}^0(\boldsymbol{X}) = \boldsymbol{X}$ is expressed as*

$$\begin{aligned} \eth g_{a,b}(\boldsymbol{X}) &= a\Bigg[\left(a^3 + 24\kappa^{(1)}a^2 + 144\kappa^{(2)}a + 192t^3\right)\frac{\sum_{i=1}^p X_i^4}{(\|\boldsymbol{X}\|^2+b)^4} - 12p\kappa^{(2)}\frac{1}{\|\boldsymbol{X}\|^2+b} \\ &\quad + 6\kappa^{(1)}(a+4\kappa^{(1)})\frac{\|\boldsymbol{X}\|^2}{(\|\boldsymbol{X}\|^2+b)^2} - 12\kappa^{(1)}(a+2\kappa^{(1)})(a+8\kappa^{(1)})\frac{\|\boldsymbol{X}\|^2}{(\|\boldsymbol{X}\|^2+b)^3} \\ &\quad + 12p\kappa^{(2)}(a+2\kappa^{(1)})\frac{1}{(\|\boldsymbol{X}\|^2+b)^2}\Bigg]. \end{aligned}$$

Theorem 3.4. *Under the model (1), the JS estimator $\boldsymbol{\delta}^{JS}_{a,b}(\boldsymbol{X})$ given by (8) dominates $\boldsymbol{\delta}^0(\boldsymbol{X})$ under quartic loss (2), for all $\boldsymbol{\theta} \in \mathbb{R}^p$, provided $p \geq 3$ and*

$$0 < a \leq 2\kappa^{(1)}(p-2) \quad \text{and} \quad b \geq max\left\{\frac{a^3 + 24\kappa^{(1)}a^2 + 12\kappa^{(2)}a(p+12) + 24p\kappa^{(3)}}{12\kappa^{(1)}(3p\kappa^{(1)} - a - 4\kappa^{(1)})}, a + 2\kappa^{(1)}\right\}.$$

The proof is similar to that of given in Theorem 3.1 with some utilities provided in the proof of Theorem 3.1 of Fourdrinier er al. (2008).

4 Examples

In this section we provide some examples of scale mixture of normal distributions to determine the superiority conditions proposed in the previous section precisely. For this purpose, we need to compute the expression $\kappa^{(i)}$ given by (5) for each distribution.

4.1 MT distribution

Suppose that $\boldsymbol{X}$ is distributed according to a MT distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\boldsymbol{I}_p$ and $\nu > 0$ degrees of freedom, denoted by $\boldsymbol{X} \sim t_p(\boldsymbol{\theta}, \boldsymbol{I}_p, \nu)$, with the following pdf

$$f(\boldsymbol{x}) = \frac{\Gamma\left(\frac{p+\nu}{2}\right)}{(\pi\nu)^{\frac{p}{2}}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{\boldsymbol{x}'\boldsymbol{x}}{\nu}\right)^{-\frac{p+\nu}{2}}.$$

The distribution is the mixture of multivariate normal distributions with the inverse gamma distribution as the weight function given by

$$G(t) = \frac{\nu^{\frac{\nu}{2}} t^{\frac{\nu}{2}-1} e^{-\frac{\nu t}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}. \tag{12}$$

By making use of the equations (5) and (12) we obtain $\kappa^{(i)} = \int_0^\infty \frac{\nu^{\frac{\nu}{2}} t^{\frac{\nu}{2}-i-1} e^{-\frac{\nu t}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\, dt = \left(\frac{\nu}{2}\right)^i \frac{\Gamma(\frac{\nu}{2}-i)}{\Gamma(\frac{\nu}{2})}$. Consequently since $\kappa^{(1)} = \frac{\nu}{\nu-2} < \infty$, $\kappa^{(2)} = \frac{\nu^2}{(\nu-2)(\nu-4)} < \infty$, $\kappa^{(-2)} = \frac{(\nu-2)(\nu-4)}{\nu^2} < \infty$, the finiteness conditions in (8) and (10) are satisfied.

4.2 MS distribution

Suppose that $\boldsymbol{X}$ is distributed according to a MS distribution with unknown location parameter $\boldsymbol{\theta}$, scale $\boldsymbol{I}_p$, denoted by $\boldsymbol{X} \sim S_p(\boldsymbol{\theta}, \boldsymbol{I}_p, 1)$, with the following pdf

$$f(\boldsymbol{x}) = \int_0^1 u^p \phi_p(u\boldsymbol{x}; u\boldsymbol{\theta}, \boldsymbol{I}_p)du = \begin{cases} \dfrac{2^{\frac{p+1}{2}-1}\gamma\left(\frac{p+1}{2}; \frac{\|\boldsymbol{x}-\boldsymbol{\theta}\|^2}{2}\right)}{(2\pi)^{\frac{p}{2}}\|\boldsymbol{x}-\boldsymbol{\theta}\|^{p+1}} & \boldsymbol{x} \neq \boldsymbol{0} \\ \frac{1}{p+1}\left(\frac{1}{2\pi}\right)^{\frac{p}{2}} & \boldsymbol{x} = \boldsymbol{0} \end{cases},$$

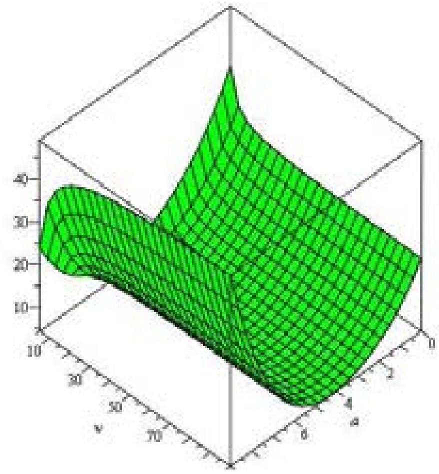

Figure 1: $R(\mathbf{0}, \boldsymbol{\delta}_a^{JS})$ **for MT model when** $p = 7$

where $\gamma(a; z) = \int_0^z t^{a-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{(-1)^k z^{a+k}}{k!(a+k)}$. See Wang and Genton (2006) for more details.

Note that the MS distribution is a scale mixture of the normal distributions (see e.g. Fang et al., 1990) and so it can be represented as:

$$\boldsymbol{X}|V = t \sim N_p(\boldsymbol{\theta}, t^{-1}\boldsymbol{I}_p), \quad V \sim U(0, 1). \tag{13}$$

Thus by making use of the equations (5) and (13) we have $\kappa^{(i)} = \int_0^1 t^{-i} dt = \frac{1}{1-i}$. One can immediately find that $\kappa^{(1)} = \infty$ and thus the required finiteness conditions are not satisfied.

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