Wrapped weighted exponential distribution

Roy, Shongkour
Department of Statistics, Jahangirnagar University
309, Bangabandhu Hall, Dhaka 1342, Bangladesh.
E- mail: sankar1604@gmail.com

Adnan, Mian
Department of Statistics, Jahangirnagar University
Dhaka 1342, Bangladesh.

1. Introduction

Weighted exponential distribution was first derived by R. D. Gupta and D. Kundu (2009). It has extensive uses in Biostatistics, and life science, etc. Since maximum important variables are axial form in bird migration study, the study of weighted exponential distribution in case of circular data can be a better perspective. Although wrapped distribution was first initiated by (Levy P.L., 1939), a few number of authors (Mardia, 972), (Rao Jammalmadaka and SenGupta, 2001), (Cohello, 2007), (Shongkour and Adnan, 2010, 2011) worked on wrapped distributions. Their works include various wrapped distributions such as wrapped exponential, wrapped gamma, wrapped chi-square, etc. However, wrapped weighted exponential distribution has never been premeditated before, although this distribution may ever be a better one to model bird orientation data.

2. Definition and derivation

A random variable X (in real line) has weighted exponential distributions if its probability density functions of form

$$f(x)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-\exp\left(-\alpha\lambda x\right)\right)\quad ;x>0,\alpha>0,\lambda>0$$

where α is shape parameter and λ is scale parameter.

Let us consider, $\theta\equiv\theta(x)=x(\operatorname{mod}2\pi)$ (2.1)

then θ is wrapped (around the circle) or wrapped weighted exponential circular random variables that for $\theta\in[0,2\pi)$ has probability density function

$$g(\theta)=\sum_{m=0}^{\infty}f_X(\theta+2m\pi) \tag{2.2}$$

$$=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum_{m=0}^{\infty}e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)})\quad ;0\le\theta\le2\pi,\alpha>0,\lambda>0$$

where the random variable θ has a wrapped weighted exponential distribution with parameter and the above pdf. as. $\theta\sim WWE(\lambda,\alpha)$.

The fig. 1 depicts the probability density function of the wrapped weighted exponential distribution.

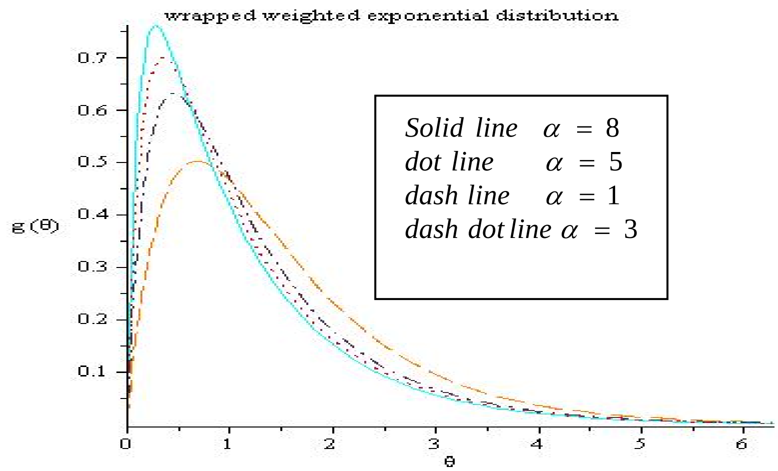

Fig. 1. Wrapped weighted exponential distribution, when $\lambda = 1$

3. Characterization of the density

The trigonometric moment of order p for a Wrapped circular distribution corresponds to the value of the characteristics function of the unwrapped r.v. X say $\phi_x(t)$ at the integer value p i.e.

$$\phi(p) = \phi_x(t) \tag{3.1}$$

In real line the characteristics function of the wrapped weighted exponential distribution

$$\phi_x(t) = E(e^{itx})$$

$$= \lambda^2 (\lambda - it)^{-1} \left(\lambda - \frac{it}{1+\alpha} \right)^{-1}$$

Hence, using (3.1) the characteristic function of the wrapped weighted exponential distribution be

$$\phi(p) = \lambda^2 (\lambda - ip)^{-1} \left(\lambda - \frac{ip}{1+\alpha} \right)^{-1}; \; p = \pm 1, \pm 2, \pm 3, \ldots \text{ Where } \quad i = (-1)^{1/2}$$

Using the fact that for $a, b, r \in R^+$, we may write $(a - ib)^{-r} = (a^2 + b^2)^{-r/2} e^{ir \arctan(\frac{b}{a})}$

$$\phi(p) = \lambda^2 (\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} e^{i \arctan \frac{p}{\lambda} \left(1 + \frac{1}{(1+\alpha)} \right)} = \rho_p e^{i \mu_p} \tag{3.2}$$

Where $\rho_p = \lambda^2 (\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}}$ *and* $\mu_p = \arctan \frac{p}{\lambda} \left(1 + \frac{1}{(1+\alpha)} \right)$

3.1. Trigonometric moments and related parameters

Using the definition trigonometric moment

$$\phi_p = \alpha_p + i\beta_p \qquad ; \quad p = \pm 1, \pm 2, \ldots$$

Therefore, the non-central trigonometric moments of the respective distribution be

$$\alpha_p = \rho_p \cos(\mu_p) \text{ and } \beta_p = \rho_p \sin(\mu_p)$$

$$\alpha_p = \lambda^2(\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} \cos\left(\arctan \frac{p}{\lambda}\left(1 + \frac{1}{(1+\alpha)} \right) \right)$$

$$\beta_p = \lambda^2(\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} \sin\left(\arctan \frac{p}{\lambda}\left(1 + \frac{1}{(1+\alpha)} \right) \right)$$

Utilizing inversion theorem of characteristic function, one can derive circular models through trigonometric moments. These trigonometric moments can be obtained using Proposition 2.1 see Rao and SenGupta (2001) . Hence, the pdf $g(\theta)$ given by

$$g(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \left(\alpha_p \cos p\theta + \beta_p \sin p\theta \right) \right]; \theta \in [0, 2\pi) \tag{3.3}$$

Using equation (3.3) we get

$$g(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \left(\lambda^2(\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} \cos\left(\arctan \frac{p}{\lambda}\left(1 + \frac{1}{(1+\alpha)} \right) \right) \cos p\theta + \lambda^2(\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} \sin\left(\arctan \frac{p}{\lambda}\left(1 + \frac{1}{(1+\alpha)} \right) \right) \sin p\theta \right) \right]$$

$$= \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \lambda^2(\lambda^2 + p^2)^{-\frac{1}{2}} \left(\lambda^2 + \left(\frac{p}{1+\alpha} \right)^2 \right)^{-\frac{1}{2}} \left(\cos p\left(\theta - \arctan \frac{1}{\lambda}\left(1 + \frac{1}{(1+\alpha)} \right) \right) \right) \right]$$

This is also alternative pdf form of wrapped weighted exponential distribution under trigonometric moment.

The graph of pdf of wrapped weighted exponential is given in Fig.1. Various characteristics of the same distribution are delivered in the following Table 1.

Theorem 1. For positive integer α if $\theta_1 \sim WWE(\alpha, \lambda)$ and $\theta_2 \sim WWE(\alpha, -\lambda)$ then

$$f_{\theta 1}(\theta) = f_{\theta 2}(2\pi - \theta) \qquad or \quad f_{\theta 2}(\theta) = f_{\theta 1}(2\pi - \theta)$$

that is positive integer α , wrapped weighted exponential distribution with symmetric rate parameters have pdf that are mirrored about origin.

Proof: For positive integer α , we may write the pdf of the linear weighted exponential distribution as

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}\left(1-\exp\left(-\alpha\lambda x\right)\right) \quad ; x>0, \alpha>0, \lambda>0$$

$$f(x) = \frac{\alpha+1}{\alpha}(-\lambda)\, e^{-(-\lambda)(-x)}\left(1-\exp\left(-\alpha(-\lambda)(-x)\right)\right)$$

$$= -f(-x;\alpha,-\lambda)$$

which shows that positive integer α ,the symmetric of $f(x;\alpha,\lambda)$ with λ >0 replaced by $-\lambda$ is a pdf of R^{-} ,mirrored form $f(x;\alpha,\lambda)$.

Using this result we may easily shows that for positive integer α for the $WWE(\alpha,\lambda)$ distribution with (λ >0),

$$f_\theta(2\pi-\theta;\alpha,-\lambda) = f_\theta(\theta;\alpha,-\lambda) \qquad ; \quad \theta\in[0,2\pi)$$

$$f_\theta(2\pi-\theta;\alpha,-\lambda) = \sum_{m=-1}^{-\infty} -f(2\pi-\theta+2m\pi;\alpha,-\lambda)$$

$$= f_\theta(\theta;\alpha,\lambda)$$

Which result shows that wrapped weighted exponential distributions with symmetric rate parameters have p.d.f's that are mirrored about origin.

4. Some important properties

In this section we show that wrapped weighted exponential distribution share some of the well-known properties of their analogs on the real line.

4.1. Hazard function: $$h(\theta) = \frac{\frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum\limits_{m=0}^{\infty} e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)})}{\left[1-\frac{\alpha+1}{\alpha}\sum\limits_{m=0}^{\infty} e^{-2m\pi\lambda}\left((1-e^{-\lambda\theta})-\frac{e^{-2\pi m\alpha\lambda}}{(1+\alpha)}\left(1-e^{-\theta(\lambda+\lambda\alpha)}\right)\right)\right]}$$

4.2. Maximum entropy property: The maximum entropy property for the wrapped weighted exponential distribution is

$$H(\theta) = -\int_0^{2\pi}\frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum_{m=0}^{\infty}e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)})\,.\ln\left[\frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum_{m=0}^{\infty}e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)})\right]d\theta$$

4.3. Infinite divisibility: An angular r.v. θ (and its probability distribution) is infinitely divisible if for any integer $n\geq 1$ there exist i.i.d. angular r.v.'s θ_1 , θ_2 ,..., θ_n such that

$$\theta_1+\theta_2+\ldots+\theta_n (\mathrm{mod}\ 2\pi) \overset{d}{=} \theta \tag{4.3}$$

Since a circular variable obtained by wrapping an infinitely divisible random variable is infinitely divisible, the infinite divisibility of the wrapped weighted exponential distributions follow from that of the linear weighted exponential distribution.

4.4. Geometric infinite divisibility: An angular r.v. θ is said to be geometric infinitely divisible if for any $q \in (0,1)$ there exist i.i.d. angular r.v.'s $\theta_1, \theta_2, \ldots, \theta_n$,

such that $$\theta_1 + \theta_2 + \ldots + \theta_{v_q} \pmod{2\pi} \overset{d}{=} \theta \qquad (4.4)$$

where v_q is the geometric distributions $p(v_q = m) = (1-q)^{m-1} q;\ m = 1, 2, 3, \ldots.$ (4.5)

5. Conclusion

In this study we consider reducing a linear variable to it's modulo 2π and using trigonometric moment to obtain expression for the pdf of wrapped weighted exponential distribution. Trigonometric moments, some common properties and also some theorems are motivated here. Modules programmed in R with circstat package, Maple-12 for performing tabulation, computations and depiction are available from the author.

Table 1: Trigonometric moment and related parameters of the WWE distribution, the entries in the last column for $\alpha = 1$

Parameter (s)	**Value (s)**	**Case $\alpha = 1$**
Trig. moments	$\alpha_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{p}{1+\alpha}\right)^2\right)^{-\frac{1}{2}} \cos\left(\arctan\frac{p}{\lambda}\left(1+\frac{1}{1+\alpha}\right)\right)$ $\beta_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{p}{1+\alpha}\right)^2\right)^{-\frac{1}{2}} \sin\left(\arctan\frac{p}{\lambda}\left(1+\frac{1}{1+\alpha}\right)\right)$	$\alpha_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\frac{p^2}{4}\right)^{-\frac{1}{2}} \cos\left(\arctan\left(\frac{1.5p}{\lambda}\right)\right)$ $\alpha_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\frac{p^2}{4}\right)^{-\frac{1}{2}} \cos\left(\arctan\left(\frac{1.5p}{\lambda}\right)\right)$
ρ_p and μ_p^0	$\rho_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{p}{1+\alpha}\right)^2\right)^{-\frac{1}{2}}$ $\mu_p^0 = \begin{cases} \arctan\frac{p}{\lambda}\left(1+\frac{1}{1+\alpha}\right) & ,\alpha < 1 \\ 2\pi + \arctan\frac{p}{\lambda}\left(1+\frac{1}{1+\alpha}\right) & ,\alpha > 1 \end{cases}$	$\rho_p = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\frac{p^2}{4}\right)^{-\frac{1}{2}}$ $\mu_p^0 = \begin{cases} \arctan\left(\frac{1.5p}{\lambda}\right) & ,\alpha < 1 \\ 2\pi + \arctan\left(\frac{1.5p}{\lambda}\right) & ,\alpha > 1 \end{cases}$
Resultant length	$\rho = \lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{1}{1+\alpha}\right)^2\right)^{-\frac{1}{2}}$	$\rho = \lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\frac{1}{4}\right)^{-\frac{1}{2}}$
Median direction	$\mu_1^0 = \begin{cases} \arctan\frac{1}{\lambda}\left(1+\frac{1}{1+\alpha}\right) & ,\alpha < 1 \\ 2\pi + \arctan\frac{1}{\lambda}\left(1+\frac{1}{1+\alpha}\right) & ,\alpha > 1 \end{cases}$	$\mu_1^0 = \begin{cases} \arctan\left(\frac{1.5}{\lambda}\right) & ,\alpha < 1 \\ 2\pi + \arctan\left(\frac{1.5}{\lambda}\right) & ,\alpha > 1 \end{cases}$

Circular variance	$V_0 = 1-\lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{1}{1+\alpha}\right)^2\right)^{-\frac{1}{2}}$	$V_0 = 1-\lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\frac{1}{4}\right)^{-\frac{1}{2}}$
Circular standard deviation	$\sigma_0 = \sqrt{-2\log\left(\lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\left(\frac{1}{1+\alpha}\right)^2\right)^{-\frac{1}{2}}\right)}$	$\sigma_0 = \sqrt{-2\log\left(\lambda^2(\lambda^2+1)^{-\frac{1}{2}}\left(\lambda^2+\frac{1}{4}\right)^{-\frac{1}{2}}\right)}$
Circular trig. moment	$\overline{\alpha_p} = \rho_p cos(\mu_p^0 - p\mu_1^0)$ and $\overline{\beta_p} = \rho_p sin(\mu_p^0 - p\mu_1^0)$	$\overline{\alpha_p} = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\frac{p^2}{4}\right)^{-\frac{1}{2}} cos\left(arctan\left(\frac{1.5p}{\lambda}\right)-p\alpha\right.$ $\overline{\beta_p} = \lambda^2(\lambda^2+p^2)^{-\frac{1}{2}}\left(\lambda^2+\frac{p^2}{4}\right)^{-\frac{1}{2}} sin\left(arctan\left(\frac{1.5p}{\lambda}\right)-p\alpha\right.$
Skewness	$\gamma_1^0 = \frac{\overline{\beta_2}}{v_0^{\frac{3}{2}}}$ where $\overline{\beta_2} = \lambda^2(\lambda^2+4)^{-\frac{1}{2}}\left(\lambda^2+\frac{4}{(1+\alpha)^2}\right)^{-\frac{1}{2}} sin\left(arctan\frac{2}{\lambda}\left(1+\frac{1}{1+\alpha}\right)-2arctan\frac{1}{\lambda}\right.$	$\gamma_1^0 = \frac{\overline{\beta_2}}{v_0^{\frac{3}{2}}}$ where $\overline{\beta_2} = \lambda^2(\lambda^2+4)^{-\frac{1}{2}}(\lambda^2+1)^{-\frac{1}{2}} sin\left(arctan\frac{3}{\lambda}-2arctan\frac{1.5}{\lambda}\right)$
Kurtosis	$\gamma_2^0 = \frac{\overline{\alpha_2}-(1-V_0)}{v_0^{\frac{3}{2}}}$ where $\overline{\alpha_2} = \lambda^2(\lambda^2+4)^{-\frac{1}{2}}\left(\lambda^2+\frac{4}{(1+\alpha)^2}\right)^{-\frac{1}{2}}$ $\left(arctan\frac{2}{\lambda}\left(1+\frac{1}{1+\alpha}\right)-2arctan\frac{1}{\lambda}\left(1+\frac{1}{1+\alpha}\right)\right)$	$\gamma_2^0 = \frac{\overline{\alpha_2}-(1-V_0)}{v_0^{\frac{3}{2}}}$ where $\overline{\alpha_2} = \lambda^2(\lambda^2+4)^{-\frac{1}{2}}(\lambda^2+1)^{-\frac{1}{2}}$ $\left(arctan\frac{3}{\lambda}-2arctan\frac{1.5}{\lambda}\right)$

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