Robust nonparametric inference for the median under a new neighborhood determined by three parameters

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1 Introduction

In robust statistical inference the departure of data from an assumed model is usually expressed by some suitably chosen neighborhood of the model distribution. The various types of neighborhoods have been used to describe the departure to date (see Huber and Ronchetti, 2009). Among them, the neighborhoods defined in terms of ε - contamination and total variation distance have been most frequently adopted in the literatures. As a combination of such two neighborhoods, Rieder (1977) introduced a neighborhood generated from a special capacity, which we call Rieder's neighborhood, and used it in his works on robust inference. Ando and Kimura (2003) proposed the (c, γ) - neighborhood which is a generalization of Rieder's neighborhood and gave its applications to robust inference.

We are concerned with the problem of constructing robust confidence intervals and tests for the median of an unknown model distribution F° when the data may be contaminated. This problem was treated under ε - contamination neighborhood by Yohai and Zamar (2004), and then under the (c, γ) - neighborhood by Ando, Kakiuchi and Kimura (2009). The purpose of this paper is to introduce a new neighborhood generated from a special capacity determined by three parameters, which we call the (c_1, c_2, γ) - neighborhood, and to study the problem under this neighborhood. The (c_1, c_2, γ) - neighborhood includes not only the (c, γ) - neighborhood but also various new neighborhoods by changing the values of the three parameters. In Section 2 we define the (c_1, c_2, γ) - neighborhood and present a list of various neighborhoods obtained as special cases. In Section 3 we give a characterization theorem of the (c_1, c_2, γ) - neighborhood. This shows that (c_1, c_2, γ) - neighborhood is natural and intuitively understandable. In Section 4 we derive robust nonparametric confidence intervals and investigate their robustness and efficiency under the (c_1, c_2, γ) - neighborhood. In Section 5 we state the result about a robustification of sign test for the median of F° under the (c_1, c_2, γ) - neighborhood. Our results refine those of Yohai and Zamar (2004) and Ando, Kakiuchi and Kimura (2009), and include theirs as special cases.

2 The $(\mathbf{c_1}, \mathbf{c_2}, \gamma)$ - neighborhood

Let $\mathbb{R}$ be the real line, $\mathscr{B}$ the Borel σ - algebra of subsets of $\mathbb{R}$ and $\mathscr{M}$ the set of all probability measures on $(\mathbb{R}, \mathscr{B})$. For some specified $F^\circ \in \mathscr{M}$ we propose the following neighborhood of F° defined as

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \leq \min\left(c_2 F^\circ\{A\} + \gamma,\ c_1 F^\circ\{A\} + 1 - c_1\right), {}^\forall A \in \mathscr{B}\}.$$

where

(2.1) $0 \leq c_1 \leq 1-\gamma \leq c_2 < \infty,\ c_1 \neq c_2$ and $0 \leq \gamma < 1$.

This neighborhood, which we call (c_1, c_2, γ) - neighborhood, includes (c, γ) - neighborhood introduced by Ando and Kimura (2003), and hence as special cases it includes Rieder's (1977) neighborhood as well as the neighborhoods defined in terms of ε - contamination and total variation distance (see Huber and Ronchetti, 2009). We can obtain various other neighborhoods by changing c_1, c_2 and γ. As easily seen, $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ is also generated from the following special capacity: Let

$$h(t) = \min\left(c_2 t + \gamma,\ c_1 t + 1 - c_1\right), \quad 0 \leq t \leq 1,$$

where $c_1,\ c_2$ and γ satisfy the condition (2.1), and let

$$v_h\{A\} = \begin{cases} h(F^\circ\{A\}), & \text{if} \quad \phi \neq {}^\forall A \in \mathscr{B}, \\ 0, & \text{if} \quad A = \phi. \end{cases}$$

Then, by Lemma 3.1 of Bednarski (1981) v_h is a special capacity, which satisfies all the conditions of Choquet's 2 - alternating capacity except condition (4) in Huber and Strassen (1973), and we have

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G \in \mathscr{M} \mid G\{A\} \leq v_h\{A\} \ \text{ for } \ {}^\forall A \in \mathscr{B}\}.$$

Thus the (c_1, c_2, γ) - neighborhood is generated from the special capacity v_h. This fact means that it has nice properties for developing minimax theory in robust inference. When $c_1 = c_2 = 1$ and $\gamma = 0$, we have $h(t) = t$ and $v_h = F^\circ$. This implies that a probability measure is a special capacity and $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = F^\circ$. As special cases we can obtain the following representative neighborhoods:

- (i) ε - contamination neighborhood: $\mathscr{P}_{c_1,1-\varepsilon,\varepsilon}(F^\circ) = \mathscr{P}_{1-\varepsilon,c_2,\varepsilon}(F^\circ)$.
- (ii) Total variation neighborhood: $\mathscr{P}_{0,1,\delta}(F^\circ)$.
- (iii) Rieder's neighborhood: $\mathscr{P}_{0,1-\varepsilon,\varepsilon+\delta}(F^\circ)$.
- (iv) (c, γ) - neighborhood: $\mathscr{P}_{0,c,\gamma}(F^\circ)$.
- (v) εt - neighborhood: $\mathscr{P}_{1-\varepsilon,1,\delta}(F^\circ)$.
- (vi) g - neighborhood: $\mathscr{P}_{c_1,c_2,0}(F^\circ)$.

Both of εt - neighborhood (v) and g - neighborhood (vi) are new, and have a special interest for us.

3 Characterization of $(\mathbf{c_1, c_2, \gamma})$ - neighborhood

Let F° be an absolutely continuous distribution function on $\mathbb{R}$ and let f° be a density function of F° (with respect to the Lebesgue measure). Also, let $\mathscr{M}_c (\subset \mathscr{M})$ be the set of all absolutely continuous distributions on $(\mathbb{R}, \mathscr{B})$. Then we obtain the following characterization theorem which shows that $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$ can be expressed in the intuitively more understandable form by using density functions.

Theorem 3.1 *For* $0 \leq c_1 \leq 1-\gamma \leq c_2 < \infty,\ c_1 \neq c_2$ *and* $0 \leq \gamma < 1$, *it holds that*

$$\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = \{G = (1-\gamma)F + \gamma K \in \mathscr{M} \mid F \in \mathscr{F}_{c_1,c_2,\gamma}(F^\circ),\ K \in \mathscr{M}\},$$

where

$$\mathscr{F}_{c_1,c_2,\gamma}(F^\circ) = \{F \in \mathscr{M}_c \mid \frac{c_1}{1-\gamma} f^\circ \leq f \leq \frac{c_2}{1-\gamma} f^\circ\}$$

and f *is a density function of* F.

Let F_L° and F_R° be the distributions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ defined by

$$(3.1)\quad F_L^\circ(x) = \begin{cases} \dfrac{c_2}{1-\gamma}F^\circ(x), & \text{if } x \le x_L, \\ \dfrac{c_1}{1-\gamma}F^\circ(x) + \left(1 - \dfrac{c_1}{1-\gamma}\right), & \text{if } x > x_L, \end{cases}$$

and

$$(3.2)\quad F_R^\circ(x) = \begin{cases} \dfrac{c_1}{1-\gamma}F^\circ(x), & \text{if } x \le x_R, \\ \dfrac{c_2}{1-\gamma}F^\circ(x) + \left(1 - \dfrac{c_2}{1-\gamma}\right), & \text{if } x > x_R, \end{cases}$$

where

$$x_L = (F^\circ)^{-1}\left(\frac{1-\gamma-c_1}{c_2-c_1}\right) \quad \text{and} \quad x_R = (F^\circ)^{-1}\left(\frac{c_2+\gamma-1}{c_2-c_1}\right),$$

respectively. Then, $F_L^\circ(x)$ and $F_R^\circ(x)$ are stochastically smallest and largest distribution functions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$, respectively, and we have $(1-\gamma)F_R^\circ(x) \le G(x) \le (1-\gamma)F_L^\circ(x) + \gamma$ for all $x \in \mathbb{R}$ and any $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$.

4 Robust nonparametric confidence intervals

Let $\boldsymbol{X}_n = (X_1, \cdots, X_n)$ be an independent and identically distributed random sample with a common distribution $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$, where F° is unknown, and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the order statistics of $\boldsymbol{X}_n$.

The purpose of this section is to construct robust nonparametric confidence intervals for the median of the unknown target distribution F° under $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$, but not for the median of the data distribution G. We assume the following two conditions:

(C1) F° is absolutely continuous with a unique median $\theta = (F^\circ)^{-1}(1/2)$.

(C2) $0 \le \gamma < 1/2, \quad c_2 < 2(1-\gamma)$.

The second condition of (C2) guarantees that $0 < F(\theta) < 1$ holds for all $F \in \mathscr{F}_{c,\gamma}(F^\circ)$, that is, the median θ lies inside the support of F. We consider the sign test statistic

$$(4.1)\quad T_{n,\theta}(\boldsymbol{X}_n) = \sum_{i=1}^{n} I_{(0,\infty)}(X_i - \theta),$$

where $I_{(0,\infty)}$ denotes the indicator function on $(0,\infty)$. Then the associated confidence interval for θ is given by

$$(4.2)\quad I_n = [X_{(k_n+1)}, X_{(n-k_n)}),$$

which is obtained by inverting the acceptance region of the sign test $k_n < T_{n,\theta}(\boldsymbol{X}_n) < n - k_n$, where k_n is the integer determined by given n, $\alpha\,(0 < \alpha < 1)$ and $\lambda\,(0 \le \lambda < 1)$ as follows:

$$(4.3)\quad k_n = k_n(n,\alpha,\lambda) = \arg\min_k |\alpha^*(n,k,\lambda) - \alpha|,$$

where $\alpha^* = \alpha^*(n,k,\lambda) = 1 - P(k < Z_n < n-k)$ with Z_n distributed as Binomial $(n, (1-\lambda)/2)$. Here we note that λ is the sum of contamination size γ and gap size $\min\{(1-\gamma)-c_1,\, c_2-(1-\gamma)\}$, that is,

$$(4.4)\quad \lambda = \min\{(1-\gamma)-c_1,\, c_2-(1-\gamma)\} + \gamma.$$

Since $0 \le \gamma \le \lambda < 1$ from (2.1), it follows that $\lambda = 0$ implies that $\gamma = 0$ and $c_1 = 1-\gamma$ or $c_2 = 1-\gamma$, and hence $\mathscr{P}_{c_1,c_2,\gamma}(F^\circ) = F^\circ$. We also note that $\lim_{n\to\infty} \alpha^*(n, k_n, \lambda) = \alpha$, that is, the exact (i.e., real) coverage probability $1-\alpha^*$ of the confidence interval (4.2) asymptotically converges to the nominal coverage probability $1-\alpha$.

The following theorem enables us to construct robust nonparametric confidence intervals over (c_1, c_2, γ) - neighborhoods and gives the result corresponding to Theorem 1 in Yohai and Zamar (2004) and Theorem 1 in Ando, Kakiuchi and Kimura (2009).

Theorem 4.1 *Let $\boldsymbol{X}_n = (X_1, \cdots, X_n)$ be a random sample from $G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)$. Then the following results hold:*

(i) $I_n = [X_{(k_n+1)}, X_{(n-k_n)})$ *has nonparametric* (c_1, c_2, γ) *- robust coverage* $1-\alpha^*(n, k_n, \lambda)$, *that is,*

$$\inf_{G\in\mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} P_G\{X_{(k_n+1)} \le \theta < X_{(n-k_n)}\} = 1-\alpha^*(n, k_n, \lambda), \tag{4.5}$$

where k_n and λ are given by (4.3) *and* (4.4), *respectively.*

(ii) *The infimum in* (4.5) *is achieved for any γ - contaminating distribution of F_L° (or F_R°) which places all its mass to the left (or the right) of θ, where F_L° and F_R° are the stochastically smallest and largest distributions in $\mathscr{F}_{c_1,c_2,\gamma}(F^\circ)$ given by* (3.1) *and* (3.2), *respectively.*

Theorem 4.1 states that the nonparametric (c_1, c_2, γ) - robust confidence interval I_n with coverage probability $1-\alpha^*(n, k_n, \lambda)$ is determined through $\lambda = \min\{(1-\gamma)-c_1,\ c_2-(1-\gamma)\}+\gamma$. We present a list of λ for the representative neighborhoods given in Section 2.

(i) ε - contamination neighborhood: $\lambda = \varepsilon$, $\quad 0 \le \varepsilon < 1/2$.

(ii) Total variation neighborhood: $\lambda = 2\delta$, $\quad 0 \le \delta < 1/2$.

(iii) Rieder's neighborhood: $\lambda = \varepsilon + 2\delta$, $\quad 0 \le \varepsilon,\ 0 \le \delta,\ \varepsilon+\delta < 1/2$.

(iv) (c, γ) - neighborhood: $\lambda = c + 2\gamma - 1$, $\quad 0 \le \gamma < 1/2,\ 1-\gamma \le c < 2(1-\gamma)$.

(v) εt - neighborhood: $\lambda = \min(\varepsilon, 2\delta)$, $\quad \delta < \varepsilon < 1,\ 0 < \delta < 1/2$.

(vi) g - neighborhood: $\lambda = \min\{1-c_1, c_2-1\}$.

When $\lambda = \varepsilon$ or $\lambda = c + 2\gamma - 1$, the confidence intervals I_n with coverage probability $1-\alpha^*(n, k_n, \lambda)$ are the same as the nonparametric ε - robust confidence interval in Yohai and Zamar (2004) or the nonparametric (c, γ) - robust confidence interval in Ando, Kakiuchi and Kimura (2009), respectively. This fact implies that the last two confidence intervals have also nonparametric (c_1, c_2, γ) - robust coverage $1-\alpha^*(n, k_n, \lambda)$ for all c_1, c_2 and γ such that $\lambda = \varepsilon$ or $\lambda = c+2\gamma-1$, respectively. When $c_1 = 1-\varepsilon$, $c_2 \ge 1+\varepsilon$ or $c_1 \le 1-\varepsilon$, $c_2 = 1+\varepsilon$, we have $\lambda = \varepsilon\,(0 \le \varepsilon < 1)$ for g - neighborhood. It should be noted that although ε - contamination neighborhood and g - neighborhood seem to be heterogeneous, both sizes λ are the same.

For some given n, α and (c_1, c_2, γ), we construct the confidence interval $I_n = [X_{(k_n+1)}, X_{(n-k_n)})$ with λ which has (c_1, c_2, γ) - robust coverage $1-\alpha^*$. The $1-\alpha$ and $1-\alpha^*$ are an nominal coverage probability and the exact coverage probability for I_n, respectively. We call (c_1, c_2, γ) the design size. In what follows, we investigate the robustness and efficiency of I_n under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ satisfying all the conditions (2.1) and (C2) for (c_1, c_2, γ) replaced by $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$. Thus we have to distinguish (c_1, c_2, γ) and $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ clearly.

The maximum asymptotic length $L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of $\{I_n\}$ under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ at F° is defined by

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \sup_{G\in\mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ)} \operatorname{essup} \limsup_{n\to\infty} (X_{(n-k_n)} - X_{(k_n+1)}),$$

where essup stands for essential supremum. The *length breakdown size* $\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of the sequence $\{I_n\}$ under the real size $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ at F° is defined as

$$\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \sup\{\lambda \mid L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} < \infty\}.$$

Theorem 4.2 *Suppose that F° has a symmetric (around θ) and unimodal density. Let $0 < \alpha < 1$ and (c, γ) be fixed and consider the sequence $\{I_n\}$ of confidence intervals $I_n = [X_{(k_n+1)}, X_{(n-k_n)})$ with k_n and λ given by (4.3) and (4.4), respectively. Then, for $0 \leq \lambda < 1 - 2\tilde{\gamma}$ the following results hold:*

(i) *If* $0 \leq \lambda < 1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}$ *and* $\tilde{c}_1 \neq 0$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(1 - \frac{1-\lambda}{2\tilde{c}_1}\right) - (F^\circ)^{-1}\left(1 - \frac{1+\lambda}{2\tilde{c}_1}\right), \quad 0 \leq \tilde{\gamma} < \frac{1}{2},$$

if $1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1} \leq \lambda < \dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1}$ *and* $\tilde{c}_1 \neq 0$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \begin{cases} (F^\circ)^{-1}\left(\dfrac{\lambda}{\tilde{c}_1} + \dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right), & 0 \leq \tilde{\gamma} < \dfrac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1}, \\ (F^\circ)^{-1}\left(1 - \dfrac{1-\lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right), & \dfrac{\tilde{c}_2(1-\tilde{c}_1)}{2\tilde{c}_2 - \tilde{c}_1} \leq \tilde{\gamma} < \dfrac{1}{2}, \end{cases}$$

and if $\dfrac{\tilde{c}_1(\tilde{c}_2 - 1 + \tilde{\gamma})}{\tilde{c}_2 - \tilde{c}_1} \leq \lambda < 1 - 2\tilde{\gamma}$, *then*

$$L\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = (F^\circ)^{-1}\left(1 - \frac{1-\lambda}{2\tilde{c}_2}\right) - (F^\circ)^{-1}\left(\frac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right), \quad 0 \leq \tilde{\gamma} < \frac{1}{2}.$$

(ii) $\lambda^*\{I_n, F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = 1 - 2\tilde{\gamma}$.

(iii) *If* $\tilde{\gamma} = \gamma$, *then the sequence* $\{I_n\}$ *has the finite maximum asymptotic length under the real size* $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$ *at* F° *if and only if*

$$\min\{(1-\gamma) - c_1,\ c_2 - (1-\gamma)\} + 3\gamma < 1.$$

(iv) *Let* $\{I'_n\}$ *be a sequence of confidence intervals* $I'_n = [A_n(\boldsymbol{X}_n), B_n(\boldsymbol{X}_n))$ *such that*

$$\inf_{G \in \mathscr{P}_{c_1,c_2,\gamma}(G^\circ)} P_G\{A_n(\boldsymbol{X}_n) \leq (G^\circ)^{-1}(1/2) < B_n(\boldsymbol{X}_n)\} = 1 - \alpha$$

for any absolutely continuous distribution G°. *Suppose that* $\lim_{n\to\infty} A_n(\boldsymbol{X}_n) = A_0$ *and* $\lim_{n\to\infty} B_n(\boldsymbol{X}_n) = B_0$ *almost surely when the sample comes from* F°. *Then it holds that*

$$A_0 \leq (F^\circ)^{-1}((1-\lambda)/2) \quad \text{and} \quad B_0 \geq (F^\circ)^{-1}((1+\lambda)/2).$$

5 Robust nonparametric tests

Let F° be a fixed unknown distribution and consider the problem of testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Let $\varphi_{n,\theta_0}(\boldsymbol{X}_n)$ be a non-randmized sign test defined by

$$(5.1)\quad \varphi_{n,\theta_0}(\boldsymbol{X}_n) = \begin{cases} 1, & \text{if } T_{n,\theta_0}(\boldsymbol{X}_n) \leq k_n \text{ or } T_{n,\theta_0}(\boldsymbol{X}_n) \geq n - k_n, \\ 0, & \text{if } k_n < T_{n,\theta_0}(\boldsymbol{X}_n) < n - k_n, \end{cases}$$

where $T_{n,\theta}(\boldsymbol{X}_n)$ is given by (4.1), and k_n and λ are given by (4.3) and (4.4), respectively.

Theorem 5.1 *A non-randmized sign test* $\varphi_{n,\theta_0}(\boldsymbol{X}_n)$ *given by* (5.1) *has nonparametric* (c_1, c_2, γ)*-robust level* $\alpha^*(n, k_n, \lambda)$*, that is,*

$$\sup_{G \in \mathscr{P}_{c_1,c_2,\gamma}(F^\circ)} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n) = 1\} = \alpha^*(n, k_n, \lambda) \quad \textit{for all } F^\circ.$$

A sequence (φ_{n,θ_0}), $n \geq n_0$, of non-randmized tests is said to have $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$*-robust power at* F° if there exists a positive real number M such that

$$\inf_{G \in \mathscr{P}_{\tilde{c}_1,\tilde{c}_2,\tilde{\gamma}}(F^\circ_\eta)} \lim_{n\to\infty} P_G\{\varphi_{n,\theta_0}(\boldsymbol{X}_n) = 1\} = 1 \quad \text{for all } |\eta| > M, \tag{5.2}$$

where $F^\circ_\eta(x) = F^\circ(x-\eta)$. The $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$*-consistency distance* $M^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of a sequence (φ_{n,θ_0}), $n \geq n_0$, of tests at F° is defined as the infimum of the set of values M for which (5.2) holds. The *power breakdown size* $\lambda^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\}$ of a sequence (φ_{n,θ_0}), $n \geq n_0$, of tests at F° is defined as the supremum of the set of values λ for which the sequence of tests is $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$-robust power at F°.

Theorem 5.2 *Suppose that* F° *has a symmetric* (*around* θ) *and unimodal density. Let* $0 < \alpha < 1$ *and consider the sequence of tests* (φ_{n,θ_0}), $n \geq n_0$, *for* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta \neq \theta_0$ *given by* (5.1). *Then, for* $0 \leq \lambda < 1 - 2\tilde{\gamma}$ *the following results hold:*

(i) *The* $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$*-consistency distance for the sequence* (φ_{n,θ_0}), $n \geq n_0$, *of tests at* F° *is*

$$M^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = \begin{cases} (F^\circ)^{-1}\left(\dfrac{1+\lambda}{2\tilde{c}_1}\right), & 0 \leq \lambda < 1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1}, \\ (F^\circ)^{-1}\left(1 - \dfrac{1-2\tilde{\gamma}-\lambda}{2\tilde{c}_2}\right), & 1 - 2\tilde{\gamma} - \dfrac{2\tilde{c}_2(1-\tilde{\gamma}-\tilde{c}_1)}{\tilde{c}_2 - \tilde{c}_1} \leq \lambda < 1 - 2\tilde{\gamma}. \end{cases}$$

(ii) $\lambda^*\{(\varphi_{n,\theta_0}), F^\circ, (\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})\} = 1 - 2\tilde{\gamma}$.

(iii) *If* $\tilde{\gamma} = \gamma$, *then the sequence* $(\varphi_{n,\theta_0}), n \geq n_0$, *of tests has* $(\tilde{c}_1, \tilde{c}_2, \tilde{\gamma})$*-robust power at* F° *if and only if*

$$\min\{(1-\gamma) - c_1,\ c_2 - (1-\gamma)\} + 3\gamma < 1.$$

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