Statistical Inference on Constant Stress Accelerated Life Tests Under Generalized Gamma Lifetime Distributions

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Introduction

In reliability study, it is important to analyze the time period over which items or systems function. Such analyses are called lifetime or failure time analyses. However, for highly reliable items, their lifetimes are usually longer under normal operating conditions. Therefore, it becomes time-consuming for observing enough failure time data or the cost of the experiment is increased. The standard method to assess highly reliable components is to test them under extreme operating conditions, referred to as accelerated life test (ALT). In such experiment, testing units are subjected to higher than usual levels of stress, such as temperature, voltage, force, humidity, and so on, so that more failures can be observed. The main purpose of the ALT is to observe more complete lifetime data under extreme conditions in order to draw statistical inference under normal conditions. ALT can be carried out using constant stress test, step-stress, or linearly increasing stress. Moreover, since there are constraints on completing a life test, some censoring procedure is commonly adopted. The most commonly used censoring schemes applied to time-step and failure-step stress tests are the Type-I censoring and Type-II censoring, respectively. Type I censoring occurs if an experiment begins with a set number of subjects or items and stops at a predetermined time. Instead, if we determine the life test or change stress level when the number of failed components reaches the pre-fixed number, this censoring scheme is called Type-II censoring. In the former case, the terminating time of the life test is fixed and the number of failure items is random; in the latter, however, the terminating time of the test is random but the number of failure observations is fixed.

The problem of ALT and making inference from Type-I and Type-II censored data has been studied by many authors. Nelson(1980) originally proposed the simple step stress ALT, in which only one change of stress occurs with a cumulative exposure model for Type-I and Type-II censored data. Miller and Nelson (1983) considered the optimal simple step-stress plans where all test components were run to failure with exponentially distributed failure time. Most authors have dealt with maximum likelihood (ML) inference and optimal test plans based on the ALT. See Meeker and Escobar (1998), Bagdonavicius and Nikulin (2002), Escobar and Meeker (2006), and the references therein. Other than the ML approach, van Dorp et al. (1996) van Dorp and Mazzuchi (2004) conducted Bayesian inference of ALTs.

Most of the papers dealing ALT tests in the literature have assumed the life time distribution of the testing item to be exponential distribution, Weibull distribution or log-normal distribution, etc. However, under more complex structure or mechanism, the above distributions may not fit the lifetime data well. On the other hand, a more general distribution which also covers all the above distributions is the generalized gamma distribution. The generalized gamma distribution was first considered by Stacy (1962). Although it covers many commonly used distributions, the exists some difficulty in calculating the maximum likelihood estimates of the underlying parameters. Lawless (1980, 1982) made a log-transformation with re-parametrization on the parameters which simplifies the calculation of the MLE and related inference. We adopt Lawless' (1982) version of the generalized gamma distribution in this paper. We will discuss n independent and identical items whose lifetime distribution is the generalized gamma distribution under K different stress levels ALT. The test is run within a fixed time interval under Type-I censoring for all stress levels. We assume that the relationship between the stress level and the location parameter of the lifetime distribution is linear. The rest of the paper is organized by the following. Section 2 introduces the notations and model assumptions. Section 3 presents the maximum likelihood inference of the underlying parameters along with the predictive issues on the mean time to failure (MTTF) and the reliability under conditions of normal use; while Section 4 applies the likelihood ratio test to identify if the ALT data are suitable to be fitted by the usual exponential distribution; or alternatively, the generalized gamma is a more appropriate model. An illustrative example based on simulation data are provided in Section 5. Finally, concluding remarks are made in Section 6.

Assumptions and Model Description

We first describe a step-stress ALT with Type-I censored data involving K stress environments $\mathbf{x} = (x_, \ldots, x_K)$. It is presumed that the ordering of the testing environments in terms of severity induces the same ordering in the associated failure rates. Suppose N identical test items are available for the step-stress ALT over experimental time $\tau > 0$ and their lifetimes are denoted by $T_1, \ldots, T_N$. The experiment is then carried out in the following manner. Letting $0 = N_0 < N_1 < \cdots < N_{K-1} < N_K = N$ be fixed, and each stress level x_k is applied to $N_{k-1}+1, \ldots, N_k$ items, for $k = 1, \ldots, K$. All the items are removed upon the experimental time τ. Let δ_i be the indicator variable to indicate if item i fails under the test, for $i = 1, \ldots, N$. If item i fails by τ, so that $T_i < \tau$, then $\delta_i = 1$; otherwise, we have $T_i = \tau$ and $\delta_i = 0$, if the ith item is still active at τ. At the end of the test, the data are collected as $\boldsymbol{d} = \{(t_i, \delta_i), i = 1, \ldots, N\}$ in which $t_i < \tau$ and $\delta_i = 1$; or $t_i = \tau$ and $\delta_i = 0$.

Under the aforementioned step-stress ALT scheme, the following three assumptions are made:

(A1) The stress level is gradually increased in k. That is

$$0 < x_0 < x_1 < x_2 < \cdots < x_K < \infty.$$

(A2) Under any stress level k, the failure times T_i of the items $i = N_{k-1}+1, \ldots, N_k$ are independent and identically distributed from a generalized gamma lifetime distribution, denoted by $GG(\beta_k, \lambda, \sigma)$, for $k = 1, 2, \cdots, K$, with the probability density function (pdf)

$$(1) \quad f_k(t) = \frac{\lambda}{\sigma t \Gamma(\lambda^{-2})}[\lambda^{-2}(e^{-\beta_k}t)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}} \; exp[-\lambda^{-2}(e^{-\beta_k}t)^{\frac{\lambda}{\sigma}}], t > 0, \beta_k > 0, \lambda > 0, \sigma > 0,$$

and the survival function

$$(2) \quad \overline{F_k}(t) = 1 - \Gamma[\lambda^{-2}(e^{-\beta_k}t)^{\frac{\lambda}{\sigma}}, \;\; \lambda^{-2}], t > 0, \beta_k > 0, \lambda > 0, \sigma > 0,$$

where $\Gamma(t;\gamma) = \int_0^t x^{\gamma-1}e^{-x}\,dx/\Gamma(\gamma)$.

(A3) The linear relationship holds between the stress variable x_k and the location parameter β_k i.e.

$$\beta_k = a + b\,x_k,$$

for $k = 1, \cdots, K$, where a, b are unknown parameters depending on the nature of the product. From (A1), the lifetime of each component is decreasing as the stress level is gradually increasing, thus $b < 0$.

Throughout the paper, define $\sum_{j=1}^{0}(\cdot) = 0$ and $\prod_{j=1}^{0}(\cdot) = 1$.

Maximum Likelihood Estimation

In this section, we derive the maximum likelihood estimates (MLEs) of the parameters underlying the ALT model described in Section 2. Given stress environments $\boldsymbol{x} = (x_1, \ldots, x_K)$, and the time duration τ, combining the pdf of the lifetime and its survival function specified by (1) and (2), we obtain the joint likelihood function of the parameter vector $\boldsymbol{\theta} = (a, b, \lambda, \sigma)$ given the data $\boldsymbol{d}$ as

$$\begin{aligned} L(\boldsymbol{\theta}|\boldsymbol{d}) &= \prod_{k=1}^{K}\prod_{i=N_{k-1}+1}^{N_k} f_k(t_i)^{\delta_i} \times [\overline{F_k}(\tau)]^{N_k-n_k} \\ &= \prod_{k=1}^{K}\prod_{i=N_{k-1}+1}^{N_k} \left\{\frac{\lambda}{\sigma t_i \Gamma(\lambda^{-2})}[\lambda^{-2}(e^{-\beta_k}t_i)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}} \exp[-\lambda^{-2}(e^{-\beta_k}t_i)^{\frac{\lambda}{\sigma}}]\right\}^{\delta_i} \\ &\quad \times \prod_{k=1}^{K}\left(1-\Gamma[\lambda^{-2}(e^{-\beta_k}\tau)^{\frac{\lambda}{\sigma}};\lambda^{-2}]\right)^{N_k-n_k}, \end{aligned} \tag{3}$$

where $\beta_k = a + bx_k$ and $n_k = \sum_{i=N_{k-1}+1}^{N_k}$, and its log-likelihood function is

$$\begin{aligned} \ell(\boldsymbol{\theta}|\mathbf{d}) &= \sum_{k=1}^{K}\sum_{i=N_{k-1}+1}^{N_k}\left\{\delta_i \log\left\{\frac{\lambda}{\sigma t_i \Gamma(\lambda^{-2})}[\lambda^{-2}(e^{-\beta_k}t_i)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}} \exp[-\lambda^{-2}(e^{-\beta_k}t_i)^{\frac{\lambda}{\sigma}}]\right\}\right\} \\ &\quad + \sum_{k=1}^{K}(N_k - n_k)\log\left\{1-\Gamma[\lambda^{-2}(e^{-\beta_k}\tau)^{\frac{\lambda}{\sigma}};\lambda^{-2}]\right\}. \end{aligned} \tag{4}$$

We apply the function 'optim()' in R software, provided by Cox et al. (2007) to find the maximum of $\boldsymbol{\theta}$ in (4), and the result $\hat{\boldsymbol{\theta}} = (\hat{a}, \hat{b}, \hat{\lambda}, \hat{\sigma}) = \underset{a,b,\lambda,\sigma}{\text{argmax}}\ \ell(a, b, \lambda, \sigma|d)$ is the MLE of $\boldsymbol{\theta}$. Under regularity conditions, as $N \to \infty$, $\hat{\boldsymbol{\theta}}'$ is asymptotically normally distributed with mean vector $\boldsymbol{\theta}'$ and variance-covariance matrix $I^{-1}(a, b, \lambda, \sigma)$ where $I(a, b, \lambda, \sigma)$ is the Fisher information matrix of $\boldsymbol{\theta}$. However, since the $I(a, b, \lambda, \sigma)$ is too complex to derive from (4), here we use the parametric bootstrap method (Efron (1979)) to approximate the standard errors of the MLEs, and the algorithm is given as follows.

Step (1). For $k = 1, \ldots, K$; and $i = N_{k-1} + 1, \ldots, N_k$, simulate $T_i^\star$ from $GG(\hat{\beta}_k, \hat{\lambda}, \hat{\sigma})$, respectively. Let $\delta_i^\star = 1$ if $0 < T_i^\star < \tau$; otherwise, $T_i^\star = \tau$ and $\delta_i^\star = 0$.

Step (2). Find the MLE of (a, b, λ, σ) based on $\boldsymbol{d}^\star = \{(t_i^\star, \delta_i^\star), i = 1, \ldots, N\}$, say $(\hat{a}^\star, \hat{b}^\star, \hat{\lambda}^\star, \hat{\sigma}^\star) = \underset{(a,b,\lambda,\sigma)}{\text{argmax}}\ \ell((a, b, \lambda, \sigma)|\boldsymbol{d}^\star)$.

Step (3). Repeat Step 1 and Step 2 B times to get $(\hat{a}^{\star(i)}, \hat{b}^{\star(i)}, \hat{\lambda}^{\star(i)}, \hat{\sigma}^{\star(i)})$, for $i = 1, 2, \cdots, B$, which form a bootstrap sample and the resulting sample standard deviations can be used to approximate the standard error of the corresponding MLE.

It is important to realize that the purpose of the step-stress ALT is not merely to estimate the unknown parameters of the model. Quite often, interest may also lie in studying the effect of the stress environments on the lifetime distribution and on the prediction of the mean time to failure (MTTF), and the reliability function of an item under the use or normal operating conditions. Let x_0 be the stress level of the use or normal environment. Then, all the above mentioned quantities of interest are indeed functions of $\boldsymbol{\theta}$. For example, the MTTF under normal condition is given by

$$(5) \quad E(T_0) = \exp(a + b\,x_0 + \frac{2\sigma}{\lambda}\ln\lambda)\frac{\Gamma(\frac{1}{\lambda^2} + \frac{\sigma}{\lambda})}{\Gamma(\frac{1}{\lambda^2})}$$

and the associated reliability function is

$$(6) \quad R(t) = P(T_0 > t) = 1 - \Gamma[\lambda^{-2}(e^{-a-b\,x_0}t)^{\frac{\lambda}{\sigma}}; \lambda^{-2}].$$

By the invariance property of MLE, we can replace $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$ into Eqs.(5) and (6) to obtain the MLEs of the corresponding quantities of interest. Furthermore, we can apply the above bootstrap sample to approximate the standard errors of the estimates to draw further statistical inference of the corresponding parameters or quantities of interest.

1 The likelihood ratio test

Note that the exponential distribution is a special case of the generalized gamma distribution. In Eq.(1), when $\lambda = \sigma = 1$, it is indeed the pdf of an exponential distribution with mean e^{β}. Therefore, under the framework defined in Sections 2 and 3, if the lifetime data T_i are obtained from the exponential distribution with means e^{β_k} under stress level x_k, for $k = 1, \ldots, K$, and $i = N_{k-1} + 1, \ldots, N_k$, then given the data $\boldsymbol{d}$, the likelihood function of (a, b) is reduced to

$$(7) \quad L(\boldsymbol{\theta}|\boldsymbol{d}) = \prod_{k=1}^{K}\prod_{i=N_{k-1}+1}^{N_k}\left\{\frac{1}{e^{\beta_k}}e^{-e^{-\beta_k}t_i}\right\}^{\delta_i} \times \left\{e^{-e^{-\beta_k}\tau}\right\}^{N_k - n_k},$$

where again $\beta_k = a + bx_k$ for $k = 1, \ldots, K$, so that the log-likelihood is

$$\ell(\boldsymbol{\theta}|\boldsymbol{d}) = \sum_{k=1}^{K}\sum_{i=N_{k-1}+1}^{N_k}\delta_i\left[-a - bx_k - e^{-(a+bx_k)}t_i\right] - (N_k - n_k)e^{-(a+bx_k)}\tau.$$

All the rest is similar to that derived in Section 3 with $\lambda = \sigma = 1$.

Exponential distribution is the most commonly used to model the lifetime distribution due to its simplicity and the memoryless property. However, in the more complex structure or mechanism, the lifetime data may not be well fitted by the exponential distribution. On the other hand, the generalized gamma distribution with three parameters is more flexible to fit complicated data. Moreover, exponential distribution is a special case of the generalized gamma distribution. It may be more appropriate to confirm the lifetime distribution through a formal statistical hypothesis testing before making further inference on the MTTF and/or reliability.

Considering the null hypothesis $H_0 : \lambda = \sigma = 1$ versus the alternative $H_a : \lambda \neq 1$ or $\sigma \neq 1$, we are interested in whether the data are from the exponential distribution or otherwise, it's from the generalized gamma distribution. We consider the likelihood ratio test with

$$\lambda(\boldsymbol{d}) = \frac{\sup_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta}|\boldsymbol{d})}{\sup_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta}|\boldsymbol{d})} = \frac{L(\hat{a}_0, \hat{b}_0, 1, 1|\boldsymbol{d})}{L(\hat{a}, \hat{b}, \hat{\lambda}, \hat{\sigma}|\boldsymbol{d})},$$

where $(\hat{a}, \hat{b}, \hat{\lambda}, \hat{\sigma})$ is the MLE based on (3) when the data are from the generalized gamma distribution; and $(\hat{a}_0, \hat{b}_0)$ is the MLE under the exponential distribution (7). As $N \to \infty$,

$$-2\ln\lambda(\boldsymbol{d}) \xrightarrow{d} \chi^2_m,$$

under H_0, i.e. the likelihood ratio statistic $-2\ln\lambda(\boldsymbol{d})$ converges in distribution to a χ^2 distribution with degrees of freedom $m = \dim(\Theta) - \dim(\Theta_0) = 2$.

Example

To illustrate how one can utilize the methodologies developed in the preceding sections, we now present an artificial example. In this example, we consider a step-stress ALT of two tress levels $x_1 = 10$ and $x_2 = 15$, each applied to 150 items over $\tau = 650$ when the lifetime of each item is simulated form the generalized gamma distribution. The stress level $x_0 = 5$ is assumed to be the normal operating environment for this purpose. Specifically, the first 150 data are simulated from $GG(6.1 - 0.06x_1, 1.3, 1.5)$ and the last 150 from $GG(6.1 - 0.06x_2, 1.3, 1.5)$, respectively and the data are censored at 650. The histograms of the simulated data are presented in Figure 1. By Eqs.(5) and (6), we know that the MTTF under the normal operating environment is 368 and the reliability at the mean lifetime is $R(368) = 0.305$.

We first test the hypothesis for $H_0 : \lambda = \sigma = 1$ versus $H_1 : \lambda \neq 1$ or $\sigma \neq 1$ based on the simulated data. Based on the likelihood ratio test given in Section 4, the resulting observed statistic is 130.016 which yields the p-value ¡0.001. In other words, the data provides strong evidence to conclude that the generalized gamma distribution indeed is the more appropriate model and the corresponding MLE's $(\hat{a}, \hat{b}, \hat{\lambda}, \hat{\sigma}) = (6.22, -0.06, 1.27, 1.54)$ which are quite accurate compared to the true parameter values. Consequently the estimate of the MTTF under normal condition is 435 with estimated standard error 159.27 based on 200 bootstrap replications, and the estimated reliability at the estimated MTTF is 0.300. On the other hand, if the data are fitted by the exponential model, it results in the estimated MTTF and the reliability estimation are 348 and 0.367 under the normal operating environment, respectively. Apparently, fitting the data with the simpler model tends to underestimate the MTTF and overestimate the reliability which may cause big loss for the producer for long term service.

Figure or Table Title

Data under stress level x_1. Data under stress level x_2.

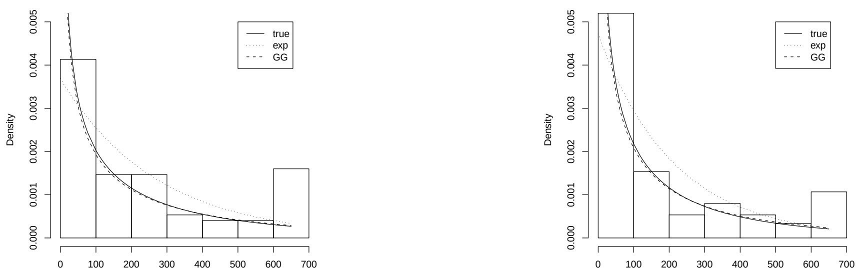

Figure 1: The histograms of the simulated data along with its true pdf (solid line), fitted pdfs by GG(dashed line) and Exp (dotted line).

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RÉSUMÉ (ABSTRACT) — optional

We will discuss the reliability analysis of constant stress accelerated life tests under generalized gamma lifetime distributions. Statistical inference on the estimation of the underlying model parameters as well as the mean life time and the reliability function will be addressed based on the maximum likelihood approach. Large sample theory will be derived for the goodness of fit of the data. Some simulation study and an illustrative example will be presented to show the appropriateness of the proposed method.