The Rank Problems of Multi-Array Data Analysis:Approaches from Complex Analysis, Algebra, Geometry and Computations

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1 Introduction

Multi-array data analysis is now rapidly developing and being applied successfully in various application fields. A multi-array datum is named a tensor. The complexity of a tensor is captured by "tensor rank" and the model complexity is captured by "maximal rank". Data analysts seek the simplest model and so the rank determination is of a vital interest for data analysis. In the determination of a maximal rank Atkinson et al. [1, 2] claimed that the maximal rank of $n \times n \times p$ tensors is less than or equal to $2n-1$. In Sumi et al. [16] we proved Atkinson and Stephens's claim over the complex number filed without assumption and over the real number field except a class of tensors. Thus we reached to some exeptional tensors; absolutely nonsingular tensors. Nonexistence of any absolutely nonsingular tensor implies that Atkinson and Stephens's bound for the maximal rank holds. Therefore, existence or nonexistence of absolutely nonsingular tensors is directly related to the claim. Sakata et al. [14] studied the existence problem for the $4 \times 4 \times 3$ tensors. In this paper we will review briefly topics mainly around absolutely nonsingular tensors for $6 \times 6 \times 3$ case from various mathematical disciplines and several new results will be given.

2 Basics of the problem

First note that throughout the paper we treat real tensors.

Definition 2.1 *An $n \times n \times p$ tensor $T = (A_1; A_2; \ldots; A_p)$ is said to be absolutely nonsingular if $\sum_{i=1}^{p} x_i A_i$ are nonsingular for all $\boldsymbol{x} = (x_1, \ldots, x_p) \neq \mathbf{0}$.*

Definition 2.2 *For an $n \times n \times p$ tensor $T = (A_1; A_2; \cdots; A_p)$, a homogeneous polynomial of degree n in $\boldsymbol{x}$,*

$$f_T(\boldsymbol{x}) = \det(\sum_{i=1}^{p} x_i A_i),$$

is called the determinant polynomial of the tensor T, (see Sakata et al. [14]).

Now the following is obvious.

Proposition 2.3 *The absolutely nonsingularity of* $T = (A_1; A_2; \cdots; A_p)$ *is equivalent to the nonzero property of the determinant polynomial* $f_T(\boldsymbol{x})$, *where the nonzero property means that* $f_T(\boldsymbol{x})$ *does not vanish for all* $\boldsymbol{x} \neq \mathbf{0}$.

Thus, the absolute nonsingularity links to the problem of the positivity of a homogeneous polynomial. The mathematical field of this topic is known as the Hilbert's 17th problem. Various methods are applicable for this problem. In Sakata et al. [14] we studied $4 \times 4 \times 3$ absolutely nonsingular tensors by Polya's theorem, which is famous in this field (see Loera and Santos [7]).

3 Reduction to an eigenvalue problem

After a definition we give another characterization of absolutely nonsingular tensors as a matrix eigenvalue problem. Note that a component matrix of a tensor is called a slice.

Definition 3.1 *Let* $IGL(n, \mathbb{R})$ *be the subset set of* $n \times n$ *nonsingular real matrices which have no real eigenvalue.*

Theorem 3.2 $T = (A_1; \ldots; A_p)$ *be an* $n \times n \times p$ *tensor without singular slices. Then,* T *is absolutely nonsingular if and only if* $x_2 A_1^{-1} A_2 + \cdots + x_p A^{-1} A_p$ *belong to* $IGL(n, \mathbb{R})$ *for all* $(x_2, \ldots, x_p) \neq \mathbf{0}$.

Proof First we prove the if part. Since $f_T(\boldsymbol{x}) = |A_1||x_1 I + x_2 A_1^{-1} A_2 + \cdots + A_1^{-1} A_p|$, if $x_1 = 0$ and $(x_2, \ldots, x_p) \neq \mathbf{0}$, $x_2 A_1^{-1} A_2 + \cdots + A_1^{-1} A_p$ should not have zero eigenvalue. On the other hand, if $x_1 \neq 0$ and $(x_2, \ldots, x_p) = \mathbf{0}$, $f_T(\boldsymbol{x}) \neq 0$ without any condition. Furthermore, if $x_1 \neq 0$ and $(x_2, \ldots, x_p) \neq \mathbf{0}$, $x_2 A_1^{-1} A_2 + \cdots + x_p A_1^{-1} A_p$ should not have any other real eigenvalue than zero. Thus, that $x_2 A_1^{-1} A_2 + \cdots + x_p A_1^{-1} A_p$ has no real eigenvalues for $(x_2, \ldots, x_p) \neq \mathbf{0}$ is the necessary condition. Conversely, it is easy to see $f_T(\boldsymbol{x})$ does not vanish for all $\boldsymbol{x} \neq \mathbf{0}$ if the condition is satified. This proves the assertion. ∎

Furthermore we show that the set of absolutely nonsingular tensors has the following algebraic structure.

Definition 3.3 *For two nonsingular matrices* A *and* B, *we define the product* $\odot$ *by*

$$A \odot B = A^{-1} B.$$

Definition 3.4 *A linear subspace* $\mathcal{S}$ *in* $IGL(n, \mathbb{R})$ *is said to be almost closed in* $IGL(n, \mathbb{R})$ *with respect to the product* $\odot$ *if* $\mathcal{S}$ *is a linear space and for* $A, B \in \mathcal{S}$, *if* $A \odot B$ *is not a constant multiple of the identity matrix,* $A \odot B \in IGL(n, \mathcal{R})$.

Then we have

Theorem 3.5 *For an absolutely nonsingular tensor* $T = (A_1; \cdots; A_p)$, *let*

$$\mathcal{A}_T = \{A | A = \sum_{i=2}^{p} x_i A_1^{-1} A_i, (x_2, \ldots, x_p) \neq \mathbf{0}\}.$$

Then $\mathcal{A}_T$ *is an almost closed linear subspace in* $IGL(n, \mathcal{R})$ *with respect to the product* $\odot$.

Proof Let $A = \sum_{i=2}^{p} x_i A_1^{-1} A_i$ and $B = \sum_{j=2}^{p} y_j A_1^{-1} A_j$. Since T is absolutely nonsingular, both A and B must be elements of $IGL(n, \mathbb{R})$ by Theorem 3.2. If $A^{-1}B$ has a real eigenvalue, say x_0, $|x_0 I + A^{-1} B| = 0$. Hence, $|x_0 A + B| = |\sum_{i=2}^{p} (x_0 x_j + y_j) A_j| = 0$. Since T is absolutely nonsingular, this means that $x_0 x_j + y_j = 0, j = 2, \ldots, p$. Then, $A^{-1} B = x_0 I$. Thus, if $A^{-1} B$ is not a constant multiple of the identity matrix, $A^{-1} B$ has no real eigenvalue and therefore it belongs $IGL(n, \mathbb{R})$. This proves the assertion. ∎

Remark 3.6 *The closedness of S in $IGL(n,\mathbb{R})$ with respect to $\odot$ does not means that S itself is closed under the product $\odot$. How is this algebraic structure connected to the existence problem of absolutely nonsingular tensors is not known at the moment.*

4 Probability of absolutely nonsingular tensors

In this section we give a simple upper bound of the probability of a tensor being nonsingular. For this we prove a lemma.

Lemma 4.1 *Let $T = (A_1; A_2; \cdots; A_p)$ be a $2n \times 2n \times p$ tensor. If there is a pair (A_i, A_j) such that $|A_i||A_j| < 0$, then T is not absolutely nonsingular.*

Proof Let $f_T(\boldsymbol{x}) = \det\left(\sum_{i=1}^{p} x_i A_i\right)$ be the determinant polynomial of T. Without loss of generality let $|A_1| > 0$ and $|A_p| < 0$. Then, setting $x_i = t^{p-i}, i = 1, 2, \ldots, p$,

$$f_T(\boldsymbol{x}) = g_T(t) = |A_1| \det\left(t^{p-1} I_{2n} + \sum_{i=2}^{p} t^{p-i} A_1^{-1} A_i\right).$$

The function $g_T(t)$ of t has the form

$$g_T(t) = t^{2n(p-1)} + \cdots + |A_1^{-1} A_p| \quad \textit{with} \quad |A_1^{-1} A_p| < 0.$$

Since $g_T(t)$ is positive for a large $|t|$ and $g_T(0) = |A_1^{-1} A_p| < 0$, $g_T(t) = 0$ has at least one real solution, and this implies that T is not absolutely nonsingular. ∎

Remark 4.2 *The function* $\det()$ *divides the space of* $2n \times 2n$ *matrices into two components of* $\det() > 0$ *and* $\det() < 0$ *with the boundary* $\det() = 0$. *Lemma 4.1 states that all the slices of an absolutely nonsingular tensor must be taken from the same component.*

Remark 4.3 *From the form of $f_T(\boldsymbol{x})$ given in Lemma 4.1 a numerically easily checkable necessary condition of the non-zero property is derived. That is, in order that T is absolutely nonsingular it is necessarily that the extended matrix*

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & -A_1^{-1} A_p \\ I & \mathbf{0} & \cdots & -A_1^{-1} A_{p-1} \\ \vdots & \ddots & \cdots & \vdots \\ \mathbf{0} & \cdots & I & -A_1^{-1} A_2 \end{pmatrix}$$

has no real eigenvalue.

From Lemma 4.1, we have

Theorem 4.4 *Let T be an randomly generated tensor where all elements of $A_i, i = 1, \ldots, p$ are independently and uniformly distributed in an interval $[-d, d]$. Then*

$$P\{T \textit{ is absolutely nonsingular}\} < \frac{1}{2^{p-1}}.$$

Proof The probability that all $|A_i|$ have the same sign is $\frac{1}{2^{p-1}}$. By Lemma 4.1 the assertion follows. ∎

The following gives another look of absolute nonsingularity.

Proposition 4.5 *Let $T = (A_1; \cdots ; A_p)$ have the property that A_i are nonsingular and $|A_i|$ have the same sign $\sigma(-$ or $+)$. Then T is absolutely nonsingular if and only if $\sum_j^p x_j A_j$ have the sign σ for all $\boldsymbol{x} \neq \mathbf{0}$.*

Remark 4.6 *The above is merely a trivial rephrase of the original non-zero property. However, we would like to stress that for absolutely nonsingular tensors, any linear combination of slices does not change the sign of the determinant, which is a quite interesting property.*

5 How to check the absolutely non-singularity

For searching absolutely nonsingular tensors we need a method that enable us to check whether a given tensor is absolutely nonsingular or not. As we have seen, this is a problem of positivity of multivariate polynomials. In fact, there are various methods for this purpose as are given below(see [6, 10, 8, 12, 18]).
(1) Polya's method(necessarily and sufficient).
(2) Solving polynomial equations (necessarily and sufficient).
(3) A sum of squares approximation(sufficient).
(4) Moment method (sufficient).
(5) Positivestellensatz (necessrily and sufficient).
Polya's method of (1) states that a homogeneous polynomial $p(\boldsymbol{x})$ is positive on the simplex $\Delta = \{x| \sum_{i=1}^p = 1, x_i \geq 0\}$ if and only if $(x_1 + \cdots + x_p)^r p(\boldsymbol{x})$ has all nonnegative coefficients for some large integer r. This may be problematic for large r.
The method (2) seeks the minimal value and shows that it is positive. For that it solves polynomial equations $\frac{\partial f_T}{\partial s}(s,t) = 0$, $\frac{\partial f_T}{\partial t}(s,t) = 0$ and checks the positivity of $f_T(\boldsymbol{x})$ at the solutions. The equations have only two variables and comparatively easily solved. However, in general, the solution is not zero dimensional(the number of solutions is not finite). For solving polynomial equations, we usually need some algebraic soft, for example, Risa Asir (see [11, 13]). However, recently, Dreesen and Moor ([3]) developed a theory solving polynomial equations by a simple linear algebra and the implementation by R is undergoing by us.
The sum of squares method of (3) is a method to seek the minimum of a polynomial $f_T(\boldsymbol{x})$ by seeking a maximal constant c such that $f_T(\boldsymbol{x}) - c$ becomes a sum of squares of polynomials. This is solved by Semi Definite Programming method. Note that R has a package of SDP. However this method is problematic for positive polynomials which cannot be expressed as a sum of squares of polynomials. It is known that such phenomena occur for $n \geq 6$.
The moment method (4) for the positivity problem of polynomials are now being developed, for example, by Lasserre [5].
Positivestellensatz of (5) states the conditions that a semi-algebraic is void. For example, to show $f_T(x,y) > 0$ on the set of $\{\boldsymbol{x}| \frac{\partial f_T}{\partial x}(x,y) = 0, \frac{\partial f_T}{\partial y}(x,y) = 0\}$ it suffices to show that the set $\{\boldsymbol{x}| f_T(x,y) \leq 0, \frac{\partial f_T}{\partial x}(x,y) = 0, \frac{\partial f_T}{\partial y}(x,y) = 0\}$ is void. There are several versions of Positivestellensatz. However, any algorithm to check the conditions seems not to be established fully.

6 Construction of absolutely nonsingular tensors

6.1 Constructible case

Here we give a method to construct $2n \times 2n \times 3$ absolutely nonsingular tensors in the case of existence of some pair of skew-symmetric matrices.

Definition 6.1 *Two matrices H_1 and H_2 is said to be independent if $|sH_1 + tH_2| \neq 0$ for $(s,t) \neq (0,0)$.*

Theorem 6.2 *Let A be any nonsingular $2n \times 2n$ matrix. If H_1 and H_2 be two nonsingular skew-symmetric matrices such that H_1 and H_2 is independent, then a tensor $T = (A; AH_1; AH_2)$ is an absolutely nonsingular tensor.*

Proof Let $f_T(x, y, z) = det(xA + yAH_1 + zAH_2)$ be the determinant polynomial of T. First we check the case of $x = 0$. Then, we have that $f_T(0, y, z) = |yAH_1 + zAH_2| = |A||yH_1 + zH_2| \neq 0$ for all $(y, z) \neq (0, 0)$ by independence assumption. Second we check the case of $x \neq 0$. We have that $f_T(x, y, z) = |xA+yAH_1+zAH_2| = |A||xI+(yH_1+zH_2)|$. Since yH_1+zH_2 is a skew-symmetric matrix, $|yH_1 + zH_2| \neq 0$ implies that $yH_1 + zH_2$ has no real eigenvalue. Therefore, $|A||xI + (yH_1 + zH_2)| \neq 0$ for all $x \neq 0$ and for all (y, z). This proves the assertion of Theorem 6.2. ∎

The next theorem gives a slightly different family of absolutely nonsingular tensors with 3 slices.

Theorem 6.3 *Let H_1 and H_2 be two nonsingular independent skew-symmetric matrices. For a positive definite or negative definite symmetric matrix S, let $A_1 = S + H_1$ and $A_2 = S + H_2$. Then, for any nonsingular matrix A, $T = (A; AA_1; AA_2)$ is absolutely nonsingular.*

Proof It suffices to show that $xA_1 + yA_2$ has no real eigenvalue for all $(x, y) \neq (0, 0)$. Note that

$$|xA_1 + yA_2| = |(x + y)S + (xH_1 + yH_2)|$$

If $x = -y \neq 0$, $|xH_1 + yH_2| \neq 0$ by independence of H_1 and H_2. For the either cases of $x > -y$ or $x < -y$, setting $s = \dfrac{x}{x+y}$ and $t = \dfrac{y}{x+y}$, it suffices to show that $|S+sH_1+tH_2| \neq 0$. Then, there is an orthogonal matrix U and a positive definite diagonal matrix D such that $|S + sH_1 + tH_2| = |D||\delta I + D^{-1/2}U(sH_1 + tH_2)U^t D^{-1/2}|$ where $\delta = \pm 1$. Since $D^{-1/2}U(xH_1 + yH_2)U^t D^{-1/2}$ is a nonsingular skew-symmetric matrix, it has no real eigenvalue, and $|\delta I + D^{-1/2}U(xH_1 + yH_2)U^t D^{-1/2}| \neq 0$ for all (s, t). This proves the assertion. ∎

6.2 Why $6 \times 6 \times 3$ absolutely nonsingular tensors are difficult to construct

For searching absolutely nonsingular tensors by using Theorem 6.2, it suffices to seek a pair of skew-symmetric matrices H_1 and H_2 such that $yH_1 + zH_2$ is independent. For the case of $n = 4$, we can find such pair of H_1 and H_2 as the elements of quaternion skew-field. However, this cannot happen for $n = 4k + 2$, stated in Theorem 6.4 below. This is one reason why it is difficult to find absolutely nonsingular $6 \times 6 \times 3$ tensors.

Theorem 6.4 *Let H_1 and H_2 be two skew-symmetric matrices of size $(4k + 2) \times (4k + 2)$. Then H_1 and H_2 can not be independent.*

Proof Consider the polynomial with a variable t, $p(t) = |H_1 + tH_2|$. Since $H_1 + tH_2$ is a skew-symmetric matrix, it's determinant is a square of a polynomial Pfaff(t), which is called the Pfaffian of $H_1 + tH_2$. That is, $p(t) = \text{Pfaff}(t)^2$. Since the degree of $p(t)$ is $4k + 2$, the degree of Pfaff(t) is $2k + 1$. Thus, Pfaff(t) vanishes in at least one real t. This proves the assertion. ∎

6.3 Nonexistence Conjecture

The problem is how to extend the assertion of Theorem 6.4 for a general pair (A, B). That is,

Conjecture 6.5 *Let $n = 4k + 2$, and let A and B be $(4k + 2) \times (4k + 2)$ matrices such that each has no real eigenvalue. Then $x_0A + y_0B$ has at least one real eigenvalue for some $(x_0, y_0) \neq (0, 0)$.*

The following supports the conjecture. The problem is to show that for any pair of $(4k+2)\times(4k+2)$ matrices A and B which have no real eigenvale, $|xI+yA+zB|\neq 0$ for all $(y,z)\neq(0,0)$. By regularizations we can treat the problem by the following two methods:
(1) To show that both $(1-s)A+sB$ and $(1-s)(-A)+sB$ have no real eigenvalues for some $0\le s\le 1$.
or
(2) To show that all of $|(1-x)I+(1+x)(1-s)A+(1+x)sB|=0$,$|(1-x)I+(1+x)(1-s)A+(1+x)sB|=0$, $|(1-x)I+(1+x)(1-s)A+(1+x)sB|=0$, and $|(1-x)I+(1+x)(1-s)A+(1+x)sB|=0$ have no solution (x,t) in the rectangle $[-1,1]\times[-1,1]$, and both $(1-s)A+sB$ and $(1-s)(-A)+sB$ do not have the eigenvalue 1. By both methods, we could not find any nonsingular pair (A,B) that does satisfy (1) or (2), for a considerably large scale of simulation experiments. Note that A can be assumed as one of the following forms and B can be assumed as a similarity transformed version of them by an arbitrary orthogonal matrix.

$$\begin{pmatrix} a & b & 0 & 0 & 0 & 0\\ -b & a & 0 & 0 & 0 & 0\\ 0 & 0 & c & d & 0 & 0\\ 0 & 0 & -d & c & 0 & 0\\ 0 & 0 & 0 & 0 & e & f\\ 0 & 0 & 0 & 0 & -f & e \end{pmatrix}, \begin{pmatrix} a & b & 0 & 0 & 0 & 0\\ -b & a & 0 & 0 & 0 & 0\\ 1 & 0 & a & b & 0 & 0\\ 0 & 1 & -b & a & 0 & 0\\ 0 & 0 & 0 & 0 & c & d\\ 0 & 0 & 0 & 0 & -d & c \end{pmatrix}, \begin{pmatrix} a & b & 0 & 0 & 0 & 0\\ -b & a & 0 & 0 & 0 & 0\\ 1 & 0 & a & b & 0 & 0\\ 0 & 1 & -b & a & 0 & 0\\ 0 & 0 & 1 & 0 & a & b\\ 0 & 0 & 0 & 1 & -b & a \end{pmatrix}$$

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7 Construction of Absolutely nonsingular tensor with $p>3$.

The method of construction of an absolutely nonsingular tensor for $p=3$ is extended to a general p. First we treat $n=4$ and $p=4$. We begin with a basic proposition about quaternions and it's matrix representation.

Proposition 7.1 *Let $\mathbb{H}_4$ be the set of quaternions $\{x*1+y*i+y*j+z*k|x,y,z\in\mathbb{R}\}$ where $i^2=j^2=k^2=ijk=-1$. Then $\mathbb{H}_4$ is a skew field and has a realization in $GL(4,\mathbb{R})$ such that*

$$1=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, i=\begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, j=\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, k=\begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this section these 4 elements are denoted by I, H_1, H_2 and H_3 respetively. In our research of absolutely nonsingular tensors, elements of the quaternion field have been consistently used (see [14, 17]).

7.1 $n=4$ and $p=4$

Proposition 7.2 *Let A be any nonsingular 4×4 matrix. Let H_1, H_2 and H_3 be the three nonsingular skew-symmetric generating matrices of the quaternion field $\mathbb{H}_4$. Then, for any nonsingular matrix A, a tensor $T=(A;AH_1;AH_2;AH_3)$ is an absolutely nonsingular tensor.*

Proof Let $f_T(x,y,z,w)=|xA+yAH_1+zAH_2+wAH_3|$ be the determinant polynomial of T. We need to show that $|xA+yAH_1+zAH_2+wAH_3|\neq 0$ for all $(x,y,z,w)\neq\mathbf{0}$. Since $|A|\neq 0$, this is equivalent to $|xI+yH_1+zH_2+wH_3|\neq 0$ for all $(x,y,z,w)\neq\mathbf{0}$. Since $xI+yH_1+zH_2+wH_3$ is a nonzero element of $\mathbb{H}_4$ for $(x,y,z,w)\neq\mathbf{0}$, it is a nonsingular matrix. This proves the assertion. ∎

7.2 $n = 4k$ and $p = 4$

Sumi et al. [17] constructed $8 \times 8 \times 8$ tensors using 7 skew-symmetric matrices. For this parameter we consider the extended skew-symmetric matrices of order $4k$ where each element is of the form $\tilde{H}_i = diag(\overbrace{H_i, H_i, \ldots, H_i}^{k})$ with $H_i \neq I \in \mathbb{H}_4$

Proposition 7.3 *Let A be any nonsingular $4k \times 4k$ matrix. Let $\tilde{H}_1$, $\tilde{H}_2$, and $\tilde{H}_3$ be nonsingular skew-symmetric matrices given in Proposition 7.1. Then a tensor $T = (A; A\tilde{H}_1; A\tilde{H}_2; A\tilde{H}_3)$ is an absolutely nonsingular tensor of size $4k \times 4k \times 4$.*

Proof The proof is the same as Proposition 7.2 and omitted. ∎

8 Equivalence and geometric invariants

Definition 8.1 *Two tensors of the same size $T_1 = (A_1; \ldots; A_p)$ and $T_2 = (B_1; \ldots; B_p)$ are said to be equivalent if*

$$\pi_P \pi_Q \pi_R T_1 = T_2$$

where π_P denotes the multiplication of each slice A_i by a nonsingular $n \times n$ matrix P from left and π_Q denotes the multiplication of each slice A_i by a nonsingular $n \times n$ matrix Q from right and π_R denotes the replacement of each slice A_i by a linear sum $\sum_{j=1}^{p} r_{ij} A_j$ where $R = (r_{ij})$ is a $p \times p$ nonsingular matrix.

Since by equivalence the rank is unchanged and therefore it is of value to check the equivalence for determining the rank of tensors. In Sakata et al. [15] , we proposed the two affine surface areas of the constant surface of the determinant polynomial;$\Omega = \{\boldsymbol{x} | f_T(\boldsymbol{x}) = 1\}$ and showed numerically its usefulness to check nonequivalence between tensors.

9 Absolutely full column rank

Miyazaki et al. [9] extended the absolute nonsingularity to rectangular tensor and gave the following definition.

Definition 9.1 *Let $T = (A_1; \ldots; A_p)$ be an $m \times n \times p$ tensor. Then if $\sum_{i=1}^{p} x_i A_i$ has rank n for all $\boldsymbol{x} \neq \boldsymbol{0}$, T is said to be an absolute full column rank tensor.*

By this tool Miyazaki et al. [9] showed several results about typical ranks, which is defined as a integer r such that there is tensors having a rank r with a positive measure.

10 Concluding remark

We reviewed a recent development of tensor rank determination problem, specially around absolutely nonsingular tensors. We gave several new results and we presented a possible root toward solving the nonexistence problem for $n = 4k + 2$. For other n, we gave several construction methods of absolutely nonsingular tensor.

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