Kurtosis and Semi-kurtosis for Portfolios Selection with Fuzzy Returns

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Abstract

The literature on portfolio analysis assumes that the securities returns are random variables with fixed expected returns and variances values (see Bachelier [1], Briec et al. [4] and Markowitz [10]). However, since investors receive efficient or inefficient information from the real world, ambiguous factors usually exist in it. Consequently, we need to consider not only random conditions but also ambiguous and subjective conditions for portfolio selection problems. A recent literature has recognized the fuzziness and the uncertainty of portfolios returns. As discussed in [6], investors can make use of fuzzy set to reflect the vagueness and ambiguity of securities (i.e. incompleteness of information due to the lack of data). Therefore, the probability theory becomes difficult to used. For example, some authors such as Tanaka and Guo [11] quantified mean and variance of a portfolio through fuzzy probability and possibility distributions, Carlsson et al.[2]-[3] used their own definitions of mean and variance of fuzzy numbers. In particular, Huang [7] quantified portfolio return and risk by the expected value and variance based on credibility measure. Recently, Huang [7] has proposed the mean-semivariance model for portfolio selection and, Li et al.[5], Kar et al.[8] introduced mean-variance-skewness for portfolio selection with fuzzy returns.

Different from Huang [7] and Li et al.[5], after recalling the definition of mean, variance, semi-variance and skewness, this paper considers the Kurtosis and semi-Kurtosis for portfolio selection with fuzzy risk factors (i.e. returns). Several empirical studies show that portfolio returns have fat tails. Generally investors would prefer a portfolio return with smaller semi-kurtosis (or Kurtosis) which indicates the leptokurtosis (fat-tails or thin-tails) when the mean value, the variance and the asymmetry are the same. Our main objective is to contribute to a sound formal foundation of statistics and finance built upon the theory of fuzzy set.

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The aim of this paper is to measure the leptokurtocity of fuzzy portfolio return by means of the two new notions of semi-kurtosis and Kurtosis. The paper is organized as follow: In Section I, we recall the notions of means, variance, semi-variance and skewness of a fuzzy variable. In Section II, we originally introduce the two notions of kurtosis and semi-kurtosis and determine some mathematical properties (which are fuzzy versions of their well-known crisp properties). We display an application in finance by establishing the mean-semivariance-skewness-semikurtosis model for a portfolio selection with fuzzy risk factors (i.e. trapezoidal risk factors).

Keywords: Fuzzy variable, Kurtosis, Semi-kurtosis, Portfolio selection.

JEL Number Classification:

1 PRELIMINARIES

1.1 Fuzzy variable and credibility

A fuzzy set A of a universe X is a mapping μ_A defined from X to $[0,1]$. If $\forall x \in X, \mu_A(x) \in \{0,1\}$, then A is a crisp set.

A fuzzy variable ξ is a fuzzy set of $\mathbb{R}$ with a membership function μ. For a real number x, $\mu(x)$ represents the possibility that ξ takes value x.

There are two usual fuzzy variables: $\xi = (a,b,c,d)$ a trapezoidal fuzzy variable ($\forall x \in]-\infty, a] \cup [d, +\infty[, \mu_\xi(x) = 0; \forall x \in [b,c], \mu_\xi(x) = 1; \forall x \in [a,b], \mu_\xi(x) = \frac{1}{b-a}x - \frac{a}{b-a}; \forall x \in [c,d], \mu_\xi(x) = \frac{1}{c-d}x - \frac{d}{c-d}$).
If $b = c$, then $\xi = (a,b,b,d) = (a,b,d)$ is a triangular fuzzy variable.
$\xi = (a,b,c,d)$ is symmetric if $\exists t \in \mathbb{R}, \forall r \in \mathbb{R}, \mu(t-r) = \mu(t+r)$.

Note that for ξ taking values in B, Zadeh [12] has defined the possibility measure of B by

$$Pos(\{\xi \in B\}) = \sup_{x \in B} \mu(x)$$

and the necessity measure of ξ by

$$Nec(\{\xi \in B\}) = 1 - \sup_{x \in B^c} \mu(x).$$

But neither, of these measures are self-dual. That reason also justified the introduction of the credibility measure by Liu ([9]).

Liu defined the credibility measure as the average of possibility measure and necessity measure as follows:

$$Cr(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x \in B} \mu(x) \;-\; \sup_{x \in B^c} \mu(x) + 1\right).$$

It is easy to show that credibility measure is self-dual. That is,

$$Cr(\{\xi \in B\}) + Cr(\{\xi \in B^c\}) = 1.$$

Let us end this Subsection by giving some notations useful throughout this paper: for a trapezoidal fuzzy variable $\xi = (a,b,c,d)$ such that $a \neq b$ and $c \neq d$, $supp(\xi) = [a,d]$ its

support, $cor(\xi) = [b, c]$ its core, l_s the length of $supp(\xi)$ and l_c the length of $cor(\xi)$. We set: $\alpha = b - a, \beta = d - c, l_s(\xi) = d - a$ and $l_c(\xi) = c - b$.
For a triangular fuzzy variable $\xi = (a, b, c)$ such that $b \neq a$ and $c \neq a$, we set: $\alpha_1 = \max\{b - a, c - b\}$ and $\gamma = \min\{b - a, c - b\}$.
$\xi = (a, b, c, d)$ is symmetric if $\alpha = \beta$, and $\xi = (a, b, c)$ is symmetric if $\alpha_1 = \gamma$.

1.2 Expected Value, Variance and Skewness of fuzzy variables

The definitions of the expected value, variance and skewness of fuzzy variables are obtained from Li et al. [5].

Definition 1. *Let ξ be a fuzzy variable. Then:*

- *its expected value $E[\xi]$ is defined as*

$$E[\xi] = e = \int_0^{+\infty} Cr\{\xi \geq r\}dr \; - \; \int_{-\infty}^{0} Cr\{\xi \leq r\}\, dr$$

  provided that at least one of the above integrals is finite.

- *its variance is defined as $V[\xi] = E[(\xi - e)^2]$.*
- *its semivariance is defined as $V^S[\xi] = E[[(\xi - e)^-]^2] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^2 \geq r\}dr$ where $(\xi - e)^-$ is a fuzzy variable defined as $(\xi - e)^- = \begin{cases} \xi - e \text{ if } \xi \leq e \\ 0 \text{ if } \xi > e \end{cases}$.*
- *its skewness is defined as $Sk[\xi] = E[(\xi - e)^3]$.*

The expected value of a trapezoidal fuzzy variable denoted $\xi = (a, b, c, d)$ is given by $E[\xi] = \frac{a+b+c+d}{4}$. Note that, expected value is one of the most important concepts of fuzzy variable, which gives the center of its distribution.
The variance of $\xi = (a, b, c, d)$ is

$$V[\xi] = -[\frac{1}{4}(l_s(\xi) + l_c(\xi))]^3(\frac{|\alpha - \beta|}{3\alpha\beta}) + \max(\frac{(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi))^3}{6\alpha \vee \beta}, 0) +$$

$$\frac{|\alpha - \beta|}{2\alpha\beta}[\frac{1}{2}l_s(\xi) - \frac{(\alpha + \beta)}{4}][\frac{1}{4}(l_s(\xi) + l_c(\xi))]^2 + \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi))^3}{6\alpha \vee \beta} - \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi))^3}{6\alpha \wedge \beta}.$$

We can easily check that if ξ is symmetric ($\alpha = \beta$), $V[\xi]$ simply becomes

$$V[\xi] = \frac{3[(c - b) + \beta]^2 + \beta^2}{24}.$$

The semi-variance of ξ is defined by

$$V^S[\xi] = \frac{1}{6(b - a)}[(\frac{e - a}{4})^3 + \min(0, (\frac{b - e}{4})^3)] + \frac{1}{6(d - c)} \max(0, (\frac{e - c}{4})^3). \quad (1)$$

The skewness of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is given by

$$Sk[\xi] = \frac{1}{8(b - a)}[(\frac{b - e}{4})^4 - (\frac{a - e}{4})^4] + \frac{1}{8(c - d)}[(\frac{c - e}{4})^4 - (\frac{d - e}{4})^4]. \quad (2)$$

Let us recall that from these results, we can deduce those of a triangular fuzzy variable.

Remark 1. The variance of ξ is used to measure the spread of its distribution about $e = E(\xi)$. Note that, variance concerns not only the part "ξ" is less than e, but also the part ξ is greater than e. If we are only interested with the first part, then we should use the concept of semi-variance.

2 Kurtosis and Semikurtosis

2.1 Definitions, examples and some first properties

Definition 2. *Let ξ be a fuzzy variable with expected value e.*

- *The kurtosis of ξ, denoted $K[\xi]$, is given by: $K[\xi] = E[(\xi - e)^4]$.*
- *The semikurtosis, $K^S[\xi]$, is given by: $K^S[\xi] = E[[(\xi - e)^-]^4] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\}dr$.*

Example 1. *For a trapezoidal fuzzy variable denoted $\xi = (a, b, c, d)$ with expected value $E[\xi] = e$, we have the following results:*

- *the kurtosis is given by*

$$K[\xi] = -[\frac{1}{4}(l_s(\xi) + l_c(\xi))]^5(\frac{|\alpha - \beta|}{5\alpha\beta}) + \max(\frac{(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi))^5}{10\alpha \vee \beta}, 0) + \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi))^5}{10\alpha \vee \beta}$$

$$\frac{|\alpha - \beta|}{2\alpha\beta}[\frac{1}{2}l_s(\xi) - \frac{(\alpha + \beta)}{4}][\frac{1}{4}(l_s(\xi) + l_c(\xi))]^4 - \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi))^5}{10\alpha \wedge \beta}$$

- *the semikurtosis is given by*

$$K^S[\xi] = \frac{1}{10(b - a)}[(\frac{e - a}{4})^5 + \min(0, (\frac{b - e}{4})^5)] + \frac{1}{10(d - c)}\max(0, (\frac{e - c}{4})^5) \quad (3)$$

It is important to notice that from these results, we can deduce those of a triangular fuzzy variable.

Proposition 1. *Let ξ be a fuzzy variable with finite expected value e, $K^S[\xi]$ and $K[\xi]$ the semi-kurtosis and kurtosis of ξ respectively. Then:*

- $0 \leq K^S[\xi] \leq K[\xi]$.
- $K^S[\xi] = 0$ *if and only if* $Cr\{\xi = e\} = 1, i.e., K[\xi] = 0$.
- *If ξ is symmetric, then $K^S[\xi] = K[\xi]$.*

Proof: 1) Let us show that $0 \leq K^S[\xi] \leq K[\xi]$.

Let $r \in \mathbb{R}$. With the definition of $(\xi - e)^-$, we have: $[(\xi - e)^-]^4 = \begin{cases} (\xi - e)^4 \text{ if } \xi \leq e \\ 0 \text{ if } \xi > e \end{cases}$. Thus we distinguish two cases as follows:

i) If $\xi(\theta) \leq e$, then $[(\xi(\theta) - e)^-]^4 = (\xi(\theta) - e)^4$. And $[(\xi(\theta) - e)^-]^4 \geq r \Leftrightarrow (\xi(\theta) - e)^4 \geq r$.

ii) If $\xi(\theta) > e$, then $[(\xi(\theta) - e)^-]^4 = 0$ and $(\xi(\theta) - e)^4 \geq [(\xi(\theta) - e)^-]^4$. Thus the inequality $[(\xi(\theta) - e)^-]^4 \geq r$ implies $(\xi(\theta) - e)^4 \geq r$. We deduce that $\forall\theta, r$, $\{\theta/[(\xi(\theta) - e)^-]^4 \geq r\} \subseteq \{\theta/(\xi(\theta) - e)^4 \geq r\}$. Since Cr is monotone, we have: $\forall r, Cr\{[(\xi - e)^-]^4 \geq r\} \leq Cr\{(\xi - e)^4 \geq r\}$. Hence $K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr \geq \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\}dr = K^S[\xi]$.

2) Let us show that $K^S[\xi] = 0$ if and only if $Cr\{\xi = e\} = 1, i.e., K[\xi] = 0$. Assume that $K[\xi] = 0$. The previous result implies that $K^S[\xi] = 0$.

Assume that $K^S[\xi] = 0$. that is, $E[[(\xi - e)^-]^4] = 0$. Since $E[[(\xi - e)^-]^4] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\}dr$, and the credibility measure Cr takes its value in $[0; 1]$, then $Cr\{[(\xi - e)^-]^4 \geq r\} = 0, \forall r > 0$. By the self-duality of Cr, we have $Cr\{[(\xi - e)^-]^4 = 0\} = 1$ and, we deduce that $Cr\{(\xi - e)^- = 0\} = 1$. Since $\xi - e = (\xi - e)^- + (\xi - e)^+$, then the previous equality implies $\xi - e = (\xi - e)^+$. And $E[(\xi - e)] = E[(\xi - e)^+] = \int_0^{+\infty} Cr\{(\xi - e)^+ \geq r\}dr = 0$. This equality implies that $Cr\{(\xi - e)^+ \geq r\} = 0, \forall r > 0$. Since Cr is self dual, we obtain $Cr\{(\xi - e)^+ = 0\} = 1$.

With $Cr\{(\xi - e)^- = 0\} = 1$ and $Cr\{(\xi - e)^+ = 0\} = 1$, we deduce $Cr\{(\xi - e) = 0\} = 1$, that is, $Cr\{\xi = e\} = 1$.
Assume that $Cr\{\xi = e\} = 1$. It is obvious to show that $K[\xi] = 0$.
3) Assume that ξ is symmetric and let us show that $K^S[\xi] = K[\xi]$.
Since $K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr$ and $K^S[\xi] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\}dr$, it suffices to show that: $Cr\{(\xi - e)^4 \geq r\} = Cr\{[(\xi - e)^-]^4 \geq r\}$. For that we distinguish two cases:
- If $r < 0$, then we have $Cr\{(\xi - e)^4 \geq r\} = Cr\{[(\xi - e)^-]^4 \geq r\} = Cr\{\Theta\} = 1$.
- If $r \geq 0$, then (with $r = r'^4$) and assume that $r' > 0$. We have $(\xi - e)^4 \geq r \Leftrightarrow (\xi - e) \in]-\infty; -r'] \cup [r'; +\infty[$, and $[(\xi - e)^-]^4 \geq r \Leftrightarrow (\xi - e)^- \in]-\infty; -r'] \cup [r'; +\infty[$. Therefore, we obtain $Cr\{(\xi - e)^4 \geq r\} = 1 - Cr\{-r' < \xi - e < r'\}, Cr\{[(\xi - e)^-]^4 \geq r\} = 1 - Cr\{-r' < (\xi - e)^- < r'\}$.
It rests to show that $Cr\{-r' < \xi - e < r'\} = Cr\{-r' < (\xi - e)^- < r'\}$.
Let μ be the membership function of $\xi - e$ and μ' be the membership function of $(\xi - e)^-$. Let us recall that $\mu' = \begin{cases} \mu \text{ if } \xi < e \\ 0 \text{ otherwise} \end{cases}$.
We have:
$Cr\{-r' < \xi - e < r'\} = \frac{1}{2}[1 + \sup_{x \in]-r';r'[} \mu(x) - \max(\sup_{x \in]-\infty;-r'[} \mu(x), \sup_{x \in]r';+\infty;[} \mu(x))] = \frac{1}{2}[1 + \sup_{x \in]-r';0[} \mu(x) - \sup_{x \in]-\infty;-r'[} \mu(x)]$. We also have $Cr\{-r' < (\xi - e)^- < r'\} = Cr\{-r' < (\xi - e)^- \leq 0\}$ since $(\xi - e)^- \leq 0$. Therefore $Cr\{-r' < (\xi - e)^- < r'\} = \frac{1}{2}[1 + \sup_{x \in]-r';0[} \mu'(x) - \max(\sup_{x \in]-\infty;-r'[} \mu'(x), \sup_{x \in]0;+\infty;[} \mu'(x))] = \frac{1}{2}[1 + \sup_{x \in]-r';0[} \mu'(x) - \sup_{x \in]-\infty;-r'[} \mu'(x)]$ since $\mu'(x) = 0, \forall x \in]0; +\infty[$.
Hence $Cr\{-r' < \xi - e < r'\} = Cr\{-r' < (\xi - e)^- < r'\}$. □

2.2 An application in finance: review, model and determinist programm

Let ξ_i be a fuzzy variable representing the return of the ith security, and let x_i be the proportion of the total capital invested in security i. In general, ξ_i is given as $\frac{(p'_i + d_i - p_i)}{p_i}$, where p_i is the closing price of the ith security at present, p'_i is the estimated closing price in the next year, and d_i is the estimated dividends during the coming year. It is clear that p'_i and d_i are unknown at present. If they are estimated as fuzzy variables, then ξ_i is also a fuzzy variable. Thereby, the portfolios $x_1, ..., x_n$ and the total return $\xi = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ are also fuzzy variables.
When minimal expected return, minimal skewness and maximal risk are given, the investors prefer a portfolio with small semi-kurtosis. Therefore, we proposed the following mean-semivariance-skewness-semikurtosis model:

$$\begin{cases} \text{minimize } K^S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq s_1 \\ V^S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \leq s_2 \\ Sk[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq s_3 \\ x_1 + x_2 + ... + x_n = 1 \\ x_i \geq 0, i = 1, 2, ..., n. \end{cases} \quad (4)$$

The first constraint of this first model ensures the expected return is no less than some target value s_1, the second one assures that risk does not exceed some given level s_2 the investor can bear, the third one assures that the skewness is no less than some target value s_3. The last two constraints imply that all the capital will be invested to n securities and short-selling is not allowed.

We now assume that $(\xi_i)_{i\in\{1,...,n\}}$ is a family of n independent triangular fuzzy variables. Thereby, based upon formulae 1, 2 and 3, we obtain the following result which display a determinist programm.

Theorem 1. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,...,n}$ be a family of n independent triangular fuzzy variables. Then the model (4) becomes the following determinist programm:*

$$\begin{cases} \min \frac{1}{10\sum_{i=1}^n x_i(b_i-a_i)}[(\frac{\sum_{i=1}^n x_i(e_i-a_i)}{4})^5 + \frac{1}{\sum_{i=1}^n x_i(b_i-d_i)}(\frac{\sum_{i=1}^n x_i(b_i-e_i)}{4})^5 \min(0, \sum_{i=1}^n x_i(b_i-e_i))] \\ \text{subject to} \\ \sum_{i=1}^n x_i(a_i+2b_i+c_i) \geq 4s_1 \\ \frac{1}{5\sum_{i=1}^n x_i(b_i-a_i)}[(\frac{\sum_{i=1}^n x_i(e_i-a_i)}{4})^3 + \frac{1}{\sum_{i=1}^n x_i(b_i-d_i)}(\frac{\sum_{i=1}^n x_i(b_i-e_i)}{4})^3 \min(0, \sum_{i=1}^n x_i(b_i-e_i))] \leq s_2 \\ (\sum_{i=1}^n x_i(c_i-a_i))^2 \sum_{i=1}^n x_i(c_i-2b_i+a_i) \geq 32s_3 \\ x_1+x_2+...+x_n = 1 \\ x_i \geq 0, i=1,2,...,n \end{cases}$$

2.3 Concluding remarks

The next step of our research will be to design a genetic algorithm integrating fuzzy simulation for our optimization model and, to apply to real financial portfolios data.

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