Testing empty intersection between the expected value of a random interval and a given interval

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1.- Introduction

The problem of testing statistical hypotheses concerning a summarizing parameter of one or more populations has been widely studied in the literature. Most of the tests developed in practice are based either on centralization or on dispersion measures, among which the population expected value and the population variances stands out. Specifically, the inclusion of the expected value of a normal random variable in a given interval has been analyzed by taking into account a multiple hypothesis test (see, Roy, 1953).

On the other hand, there exist some real-life situations in which experimental data show a certain inherent lack of precision. This is the case, for instance, of the subjective perception of the length of an object, the tides fluctuation or the temperature along a specific day. To sum up these observations in a unique value would entail a loss of information that can be avoided by using compact intervals.

Random intervals have been shown to be useful in handling this kind of random experiments, associating compact intervals with experimental outcomes. In this context, some studies concerning the expected value of a random interval have been developed as, for instance, the inclusion of the expected value of a random interval in a previously fixed interval (see Ramos-Guajardo et al., 2010). It should be remarked that the expected value in this case is understood in the sense of the Aumann integral, which leads to a compact interval.

Other situations require to analyze whether or not the mean of a random interval and a given interval are disjoint. For instance, this is the case of the mean of the perceptions provided by the experts about the quality of a specific product and a prefixed interval representing bad quality of the product. In this work a hypothesis test to deal with these situations is proposed by using both asymptotic and bootstrap techniques. Both approaches have been shown to be useful for testing hypotheses tests when imprecise values are considered (see, for instance, Montenegro et al., 2004; González-Rodríguez et al., 2006).

In Section 2, some preliminaries about random intervals are gathered. The asymptotic and bootstrap procedures for testing the null hypothesis of the expected value of a random interval and a fixed interval not overlapping are presented in Section 3. In Section 4, some simulations are carried out to show the empirical behaviour of the test as well as an application in a real-life situation. Finally, some concluding remarks and open problems are presented in Section 6.

2.- Preliminaries

Let $K_c(\mathbb{R})$ be the class of the nonempty convex compact subsets of $\mathbb{R}$. An element $A \in K_c(\mathbb{R})$ can be represented in two different ways. One considers the infimum and the supremum of the interval ($A = [\inf A, \sup A]$) satisfying the order constraint $\inf A \leq \sup A$. Due to the difficulties operating with order restrictions, it is often more convenient to work with the *mid/spr parametrization of* A given by $(\mathrm{mid}\,, \mathrm{spr}\,) \in \mathbb{R} \times \mathbb{R}^+$ defined as:

$$\mathrm{mid}\, A = (\sup A + \inf A)/2 \quad \text{and} \quad \mathrm{spr}\, A = (\sup A - \inf A)/2.$$

The class $K_c(\mathbb{R})$ can be endowed with the Minkowski addition and the product by a scalar, so that for all intervals $A, B \in K_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$

$$A + B = [\mathrm{mid}\, A + \mathrm{mid}\, B - (\mathrm{spr}\, A + \mathrm{spr}\, B), \mathrm{mid}\, A + \mathrm{mid}\, B - (\mathrm{spr}\, A + \mathrm{spr}\, B)],$$

$$\lambda A = [\lambda \mathrm{mid}\, A - \lambda \mathrm{spr}\, A, \lambda \mathrm{mid}\, A + \lambda \mathrm{spr}\, A].$$

It should be remarked that the space $[K_c(\mathbb{R}), +, \cdot]$ is not linear due to the lack of a symmetric element with respect to the Minkowski addition. To overcome this problem it is useful to consider the so-called Hukuhara difference $A -_H B$, which is defined as the set difference C, provided that $C \in K_c(\mathbb{R})$, so that $A = B + C$. If $A, B \in K_c(\mathbb{R})$, it is is easy to show that $A -_H B$ exists if and only if $\mathrm{spr}\, B \leq \mathrm{spr}\, A$.

If $(\Omega, \mathcal{A}, P)$ is a probability space, a *random interval* is usually defined as a Borel measurable mapping $X : \Omega \longrightarrow K_c(\mathbb{R})$ with respect to the σ-field generated by the topology induced by the well-known Hausdorff metric d_H on $K_c(\mathbb{R})$. Equivalently, X is a random interval if and only if $\mathrm{mid}\, X$ and $\mathrm{spr}\, X$ are real random variables.

On the other hand, if X is a random interval verifying that $E(|X|) < \infty$ (with $|X|(w) = \sup\{|x| \text{ s.t. } x \in X(w) \text{ for } \omega \in \Omega\}$), then the expected value of X in Kudo-Aumann's sense (see, e.g., Aumann, 1965) is given by $E(X) = \Big\{E(f) \,|\, f : \Omega \to \mathbb{R}\,,\, f \in L^1((\Omega, \mathcal{A}, P))\,,\, f \in X \; a.s.[P]\Big\}$. Equivalently, $E(X)$ can be expressed as $E(X) = [E(\mathrm{mid}\, X) - E(\mathrm{spr}\, X), E(\mathrm{mid}\, X) + E(\mathrm{spr}\, X)]$. In addition, the expected value of a random interval is linear and it is coherent with the arithmetic considered for finite populations and in the sense of the Strong Law of Large Numbers (see Arstein & Vitale, 1975).

Let $A, B \in \mathcal{K}_c(\mathbb{R})$. The *length* or *measure of* A is given by $m(A) = 2\text{spr}\, A$, and the *length* or *measure of the intersection between* A *and* B is given by the following expression (see Shawe-Taylor & Cristianini, 2004):

$$m(A \cap B) = \max\left\{0, \min\left\{2\text{spr}A, 2\text{spr}B, \text{spr}A + \text{spr}B - |\text{mid}A - \text{mid}B|\right\}\right\}.$$

3.- Hypothesis test about the non-overlapping of the expected value of a random interval and a given interval

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $X : \Omega \longrightarrow \mathcal{K}_c(\mathbb{R})$ be a random interval such that $\text{spr}\, E(X) > 0$ and $A \in \mathcal{K}_c(\mathbb{R})$ be fixed such that $\text{spr}\, A > 0$. The aim is to test the hypotheses

$$H_0 : E(X) \cap A = \emptyset \quad vs \quad H_1 : E(X) \cap A \neq \emptyset,$$

that can be also written as

$$H_0 : m(E(X) \cap A) = 0 \quad vs \quad H_1 : m(E(X) \cap A) \neq 0,$$

or, equivalently,

(1) $\quad H_0 : |\text{mid}\, E(X) - \text{mid}\, A| + \text{spr}\, E(X) \geq \text{spr}\, A \quad vs \quad H_1 : |\text{mid}\, E(X) - \text{mid}\, A| + \text{spr}\, E(X) < \text{spr}\, A.$

Let $X_1, \ldots, X_n$ be a simple random sample drawn from X. The sample mean associated with $X_1, \ldots, X_n$ is defined as usual on the basis of the interval arithmetic. To solve the proposed test, the following test statistic is used:

(2) $$T_n = \sqrt{n}\left(\left|\text{mid}\, \overline{X_n} - \text{mid}\, A\right| - \text{spr}\, \overline{X_n} - \text{spr}\, A\right).$$

Consider the class $\mathcal{P} = \left\{Y : \Omega \to \mathcal{K}_c(\mathbb{R}), |\, \sigma^2_{\text{mid}\, Y} < \infty \;\wedge\; 0 < \sigma^2_{\text{spr}\, Y} < \infty\right\}$. Then, the convergence of the previous statistics is given in the following result.

Theorem 3.1. *For all $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be n random intervals which are independent and identically distributed as X associated with the probability space $(\Omega, \mathcal{A}, P)$. Let T_n be defined as in (2). If $X \in \mathcal{P}$ and $\rho \in [0, 1]$, then*

i.1) *Whenever* $\text{mid}\, E(X) - \text{mid}\, A - \text{spr}\, E(X) = \text{spr}\, A$, *it is satisfied that*

(3) $$T_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\text{mid}\, X - \text{spr}\, X}) \quad as\ n \to \infty.$$

i.2) *Whenever* $-\text{mid}\, E(X) + \text{mid}\, A - \text{spr}\, E(X) = \text{spr}\, A$, *it is satisfied that*

(4) $$T_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\text{mid}\, X + \text{spr}\, X}) \quad as\ n \to \infty.$$

ii) *Let k_ρ be the maximum between the percentiles of order 100ρ of the asymptotic distributions (3) and (4). If the null hypothesis in (1) is fulfilled, then*

$$\limsup_{n \to \infty} P\left(T_n < k_\rho\right) \leq \rho$$

and the equality holds either if $\text{mid}\, E(X) - \text{mid}\, A - \text{spr}\, E(X) = \text{spr}\, A$ *or* $-\text{mid}\, E(X) + \text{mid}\, A - \text{spr}\, E(X) = \text{spr}\, A$. *As a consequence, the test that rejects H_0 in (1) whenever $T_n < k_\rho$ is asymptotically correct.*

Due to the difficulties existing on handling the limit distribution of the test statistic, the employment of other techniques is required in practice. Bootstrap procedures have been shown to be useful in solving hypotheses tests when imprecise values are involved (see, for instance, Montenegro *et al.*, 2004; Gil *et al.*, 2006; González-Rodríguez *et al.*, 2006). Specifically, a residual bootstrap is employed as follows:

Let $X : \Omega \longrightarrow K_c(\mathbb{R})$ be a random interval such that $\operatorname{spr} E(X) > 0$, $A \in K_c(\mathbb{R})$ be fixed such that $\operatorname{spr} A > 0$ and $\{X_i\}_{i=1}^n$ be a simple random sample drawn from X. The following bootstrap populations are considered:

$$\left\{Y_i^1\right\}_{i=1}^n = \left\{X_i + \operatorname{mid} A - \overline{X_n} + \operatorname{spr} A\right\}_{i=1}^n \quad \text{and} \quad \left\{Y_i^2\right\}_{i=1}^n = \left\{X_i + \operatorname{mid} A - \overline{X_n} - \operatorname{spr} A\right\}_{i=1}^n .$$

It is easy to see that $\operatorname{mid} \overline{Y_n^1} - \operatorname{mid} A - \operatorname{spr} \overline{Y_n^1} = \operatorname{spr} A$ and $-\operatorname{mid} \overline{Y_n^2} + \operatorname{mid} A - \operatorname{spr} \overline{Y_n^2} = \operatorname{spr} A$, which are the two worst cases under the null hypothesis as it was indicated in Theorem 3.1. Suppose now that $\{X_i^*\}_{i=1}^n$, $\{Y_i^{1*}\}_{i=1}^n$ and $\{Y_i^{2*}\}_{i=1}^n$ are bootstrap samples drawn from the previous ones. Therefore, the following bootstrap statistics are taken into account:

$$\begin{aligned} T_n^{1^*} &= \sqrt{n}\left(\left|\operatorname{mid} \overline{Y_n^{1*}} - \operatorname{mid} A\right| - \operatorname{spr} \overline{Y_n^{1*}} - \operatorname{spr} A\right) \\ &= \sqrt{n}\left(\left|\operatorname{mid} \overline{X_n^*} - \operatorname{mid} \overline{X_n} + \operatorname{spr} A\right| - \left(\operatorname{spr} \overline{X_n^*} - \operatorname{spr} \overline{X_n}\right) - \operatorname{spr} A\right), \end{aligned} \tag{5}$$

$$\begin{aligned} T_n^{2^*} &= \sqrt{n}\left(\left|\operatorname{mid} \overline{Y_n^{2*}} - \operatorname{mid} A\right| - \operatorname{spr} \overline{Y_n^{2*}} - \operatorname{spr} A\right) \\ &= \sqrt{n}\left(\left|\operatorname{mid} \overline{X_n^*} - \operatorname{mid} \overline{X_n} - \operatorname{spr} A\right| - \left(\operatorname{spr} \overline{X_n^*} - \operatorname{spr} \overline{X_n}\right) - \operatorname{spr} A\right). \end{aligned} \tag{6}$$

In the following result it is shown that the proposed bootstrap is consistent.

Theorem 3.2. *Let $T_n^{1^*}$ and $T_n^{2^*}$ be defined as in (5) and (6). If $X \in \mathcal{P}$ and $\rho \in [0,1]$, then*

i.1) *Whenever* $\operatorname{mid} E(X) - \operatorname{mid} A - \operatorname{spr} E(X) = \operatorname{spr} A$, *it is satisfied that*

$$T_n^{1^*} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\operatorname{mid} X - \operatorname{spr} X}) \ \ a.s. - [P]. \tag{7}$$

i.2) *Whenever* $-\operatorname{mid} E(X) + \operatorname{mid} A - \operatorname{spr} E(X) = \operatorname{spr} A$, *it is satisfied that*

$$T_n^{2^*} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\operatorname{mid} X + \operatorname{spr} X}) \ \ a.s. - [P]. \tag{8}$$

ii) *Let k_ρ^* be the maximum between the 100ρ-percentiles of the asymptotic distributions (7) and (8). If H_0 in (1) is fulfilled, then*

$$\limsup_{n\to\infty} P\left(T_n < k_\rho^*\right) \leq \rho \quad c.s. - [P]$$

and the equality holds either if $\operatorname{mid} E(X) - \operatorname{mid} A - \operatorname{spr} E(X) = \operatorname{spr} A$ *or* $-\operatorname{mid} E(X) + \operatorname{mid} A - \operatorname{spr} E(X) = \operatorname{spr} A$. *As a consequence, the test that rejects H_0 in (1) whenever $T_n < k_\rho^*$ is asymptotically correct.*

Algorithm for the bootstrap test

Step 1. Compute the value of the statistic T_n defined in (2) for $\{\mathcal{X}_i\}_{i=1}^n$.

Step 2. Obtain a bootstrap sample from $\{\mathcal{X}_i\}_{i=1}^n$ and compute the value of the bootstrap statistics $T_n^{1^*}$ and $T_n^{2^*}$ defined in (5) and (6), respectively.

Step 3. Repeat *Step 2.* a large number B of times to get a set of B values of the bootstrap statistics, denoted by E_1^* and E_2^*.

Step 4. Compute the bootstrap p-value as the maximum of the proportion of values in the sets E_1^* and E_2^* which are greater than T_n.

4.- Simulations and real-life application

To show the performance of the proposed test, some simulations are carried out. Specifically, the following hypothesis test is analyzed:

$$H_0 : E(X) \cap C = \emptyset \quad vs \quad E(X) \cap C \neq \emptyset, \quad \text{with } C = [3, 5].$$

Given a random interval X, two different situations are considered depending on the distributions chosen for the mid and the spread of X, namely,

Case 1. $\text{mid}\, X \equiv \mathcal{N}(1, 5)$ and $\text{spr}\, X \equiv \chi_2^2$.

Case 2. $\text{mid}\, X \equiv U(-4, 6)$ and $\text{spr}\, X \equiv \chi_2^2$.

It is easy to show that $E(X) = [-1, 3]$ in both cases and therefore the null hypothesis is fulfilled. Simple random samples of different sizes have been drawn from X. In addition, 10,000 simulations at different nominal significance levels have been carried out, as well as 1,000 bootstrap replications, which entails a sampling error of .195% for $\rho = .01$, .427% for $\rho = .05$ and .588% for $\rho = .1$, with a 95% confidence level. The results are gathered in Table 1.

	Case 1.			Case 2.		
$n \diagdown 100 \cdot \rho$	1	5	10	1	5	10
10	14.79	22.04	27.74	4.73	11.00	16.33
30	7.19	13.22	18.32	1.72	6.15	10.79
50	4.22	9.91	14.76	1.19	5.06	10.08
100	2.14	6.26	10.73	1.08	4.99	10.16
200	1.06	5.25	10.56	0.89	5.03	9.94

Table 1: Empirical size of the boostrap test for the empty intersection between $E(X)$ and C

Table 1 shows that the percentage of rejections of the null hypothesis is quite close to the corresponding nominal significance levels at sample sizes greater than or equal to 100. Therefore, the sample size needed to obtain suitable results is moderate. As an open research direction, it could be interesting to introduce the estimate of the covariance matrix that appears in the limit distribution of the statistic in order to improve the results obtained.
On the other hand, a real-life situation is analyzed. Some studies about the quality of a specific kind of cheese from Asturias (Spain), named Gamonedo blue cheese (see, for instance, González de Llano et al., 1992), have been recently carried out in the LILA institute. The studies were centered on the aspect, the smell quality, the flavor quality, etc, of the Gamonedo cheese. The perceptions of the experts about these features were given by means of trapezoidal fuzzy data. In this case, the study is simplified by considering only the corresponding upper interval of the trapezoids and one expert.

Specifically, the smell quality of the Gamonedo cheese is analyzed. Therefore, the random interval to consider is $X \equiv$"perception of the expert about the smell quality of each cheese". The aim is to check whether the mean of the opinions of the expert concerning the smell of the cheese is not at all included in the interval $[0, 60]$ (which represents bad quality of the cheese). Therefore, the following test is proposed:

$$H_0 : E(X) \cap [0, 60] = \emptyset \;\; vs \;\; E(X) \cap [0, 60] \neq \emptyset.$$

A random sample of size 20 has been considered and the bootstrap approach proposed in this work has been applied (with 1,000 bootstrap replications), and a p-value of $p = .9543$ has been obtained, what implies that that the null hypothesis is not rejected at the usual nominal significance levels. Therefore, the expected value of the perceptions of the expert about the smell quality of the cheese cannot be considered to be 'bad'.

5.- Concluding remarks

Asymptotic and bootstrap hypothesis tests have been proposed in this work to deal with the problem of analyzing the empty intersection between the expected value of a random interval and a fixed interval. Simulations have shown a good empirical behavior of the test for moderate sample sizes.

Nevertheless, it could be interesting to estimate the covariance matrix involved in the limit distribution of the test statistic as well as to analyze the power function of the proposed test. Furthermore, the one-sample test when the inclusion is only partial would be appealing to be discussed. Finally, the results developed in this work can be extended to tests in $\mathbb{R}^p$ (with $p > 1$) as well as to random elements for which values are fuzzy.

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