RELIABILITY STUDIES OF BIVARIATE LOG-NORMAL DISTRIBUTION

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Abstract In this paper, we study the bivariate log-normal distribution from a reliability point of view. The monotonicity of the hazard rates of the univariate as well as the conditional distributions is discussed. The probability distributions, in the case of series and parallel systems, are derived and the monotonicity of their failure rates is discussed.

1. INTRODUCTION

It has been observed that, in many practical problems, the log-normal distribution is widely used to model positive random variable that exhibits skewness. The log-normal distribution is commonly used in medicine and economics, where the basic process under consideration leads to a skewed distribution. Zou et al. (2009) comment that the log-normal distribution may be used to approximate right skewed data arising in a range of scientific enquiries, see also Limpert et al. (2001).

In order to test the equality of means of health care costs in a paired design, Zhou et al. (2001) showed that a bivariate log-normal distribution is appropriate to model the outpatient costs, six months before and after the Medicaid policy change in the state of Indiana. Hawkins (2002) discussed an example dealing with 56 assay pairs of cyclosporin from blood samples of organ transplant recipients obtained by two different methods. In this connection, assuming the bivariate log-normal distribution, Zhou et al. (2001) presented five tests for the equality of two log-normal means.

In this paper, we are interested in studying the class of bivariate log-normal distributions from a reliability point of view. More specifically, we study the association between the variables and obtain conditions for which this class of distributions is TP_2 (totally positive of order 2)or RR_2 (reverse rule of order 2). This enables us to study the dependence properties of the model. We study the hazard components of the hazard gradient in the sense of Johnson and Kotz (1975) and their monotonic structure. In this connection, the conditional distribution of X given $Y > y$ is found to be log-skew normal, a class not widely studied in the literature. An association measure $\theta(x, y)$ defined by Oakes (1989), is investigated for this class of bivariate distributions . Some of the results presented here are general and would be useful in studying the association in other classes of bivariate distributions.

2. THE MODEL

A random variable (X, Y) is said to have bivariate log-normal distribution if $(\ln X, \ln Y)$ has a bivariate normal distribution. It is clear that the marginal and conditional distributions are univariate log-normal.

For the bivariate normal distribution, it is well known that the joint density is TP_2 according as $\rho > (<)0$, see for example Joe (1997).

We now state the following result

Theorem 1. *Suppose the random variable (X, Y) has the $TP_2(RR_2)$ property. Let $h(.)$ be an increasing function. Then $(h(X), h(Y))$ has the $TP_2(RR_2)$ property.*

Proof. See Shaked (1977). ■

3. Bivariate Log-normal Distribution

Suppose (X, Y) has a bivariate log-normal distribution so that $(\ln X, \ln Y)$ has a bivariate normal distribution. It is evident that the conditional distribution of X given $Y = y$ or of Y given $X = x$ is univariate log-normal. Therefore, their reliability functions can be easily obtained by using the formulas for log-normal distribution and appropriate values of the parameters. The monotonicity of the failure rate and of the mean residual life function will follow the same pattern as of the univariate log-normal distribution.

To study the monotonicity of the failure rate of the conditional distribution of X given $Y = y$ as a function of y, we employ the following result due to Shaked (1977).

Lemma 2. *If $f(x, y)$ is $TP_2(RR_2)$, the conditional failure rate of X given $Y = y$ is decreasing(increasing) in y.*

Using the above result, we conclude that the failure rate of the conditional distribution of X given $Y = y$ is decreasing (increasing) in y according as $\rho > (<)0$.

We now discuss the conditional distribution of X given $Y > y$.

Conditional distribution of $X|Y > y$

Suppose $(X, Y) \sim BVLN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Then the conditional density of X given $Y > y$ is given by

$$
\begin{aligned}
f_{X|Y>y}(x|Y > y) &= -\frac{\partial}{\partial x}P(X > x|Y > y) \\
&= \int_y^\infty f(x,v)dv/P(Y > y) \\
&= \frac{\int_y^\infty \frac{1}{2\pi\sigma_1\sigma_2 xv\sqrt{1-\rho^2}} \exp\{\frac{-1}{2(1-\rho^2)}[(\frac{\ln x-\mu_1}{\sigma_1})^2 - 2\rho(\frac{\ln x-\mu_1}{\sigma_1})(\frac{\ln v-\mu_2}{\sigma_2}) + (\frac{\ln v-\mu_2}{\sigma_2})^2]\}dv}{P(Y > y)} \\
&= \frac{\frac{1}{x\sigma_1}\phi(\frac{\ln x-\mu_1}{\sigma_1})[1-\Phi(\frac{\ln y-[\mu_2+(\rho\sigma_2/\sigma_1)(\ln x-\mu_1)]}{\sigma_2\sqrt{1-\rho^2}})]}{1-\Phi(\frac{\ln y-\mu_2}{\sigma_2})}.
\end{aligned}
$$

Special case

When $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$, the above reduces to

$$f_{X|Y>y}(x|Y > y) = f_X(x)[1 - \Phi(\frac{\ln y - \rho \ln x}{\sqrt{1-\rho^2}})]/P(Y > y), \tag{3.1}$$

where $f_X(x)$ is the marginal density function of a standard log-normal distribution. Thus X given $Y > y$ has a log-skew normal distribution. Note that this result is true for standard as well as for non standard bivariate log-normal vectors, see Arnold (2009).

Hazard components and their monotonicity

In order to study the hazard components and their monotonicity, it is enough to consider the standard bivariate log-normal model.

The hazard rate of the conditional distribution of X given $Y > y$ is given by

$$
\begin{aligned}
h_1(x,y) &= -\frac{d}{dx}\ln \overline{F}_\rho(x,y) \qquad (3.2) \\
&= \int_y^\infty f(x,v)dv/\overline{F}(x,y) \\
&= f_X(x)[1-\Phi(\frac{\ln y - \rho\ln x}{\sqrt{1-\rho^2}})]/\overline{F}_\rho(x,y),
\end{aligned}
$$

. Using equation (3.1), the distribution of X given $Y > y$ is log-skew normal, its hazard rate is of the type U.

In a similar manner, we can obtain the conditional density of Y given $X > x$ and the hazard component $h_2(x,y)$.

The monotonicity of the conditional hazard of X given $Y > y$ as a function of y can be determined by employing the following result due to Shaked (1977).

Lemma 3. *If $f(x,y)$ is $TP_2(RR_2)$, the conditional failure rate of X given $Y > y$ is decreasing(increasing) in y*

As noticed before, the bivariate log-normal distribution is $TP_2(RR_2)$ according as $\rho > (<)0$. Thus, in the case of bivariate log-normal distribution $h_1(x, y)$ is decreasing (increasing) in y according as $\rho > (<)0$. Similar statement can be made regarding the other hazard component.

4. SERIES AND PARALLEL SYSTEMS

In this section, we shall obtain the density functions of $T_1 = Min(X, Y)$ and $T_2 = Max(X, Y)$. Also we study the monotonicity of the failure rates of T_1 and T_2.

We know that for any bivariate vector (X, Y) , the density functions of T_1 and T_2 are given by

$$f_{T_1}(t) = f_X(t)P(Y > t|X = t) + f_Y(t)P(X > t|Y = t) \tag{4.1}$$

and

$$f_{T_2}(t) = f_X(t)P(Y < t|X = t) + f_Y(t)P(X < t|Y = t), \tag{4.2}$$

see Gupta and Gupta (2001).

For the bivariate log-normal distribution, using (4.1), it can be verified that

$$\begin{aligned} f_{T_1}(t) &= \frac{1}{\sigma_1 t}\phi(\frac{\ln t - \mu_1}{\sigma_1})[1 - \Phi(\frac{\ln t(\frac{1}{\sigma_2} - \frac{\rho}{\sigma_1})}{\sqrt{1-\rho^2}} + \frac{\frac{\mu_1\rho}{\sigma_1} - \frac{\mu_2}{\sigma_2}}{\sqrt{1-\rho^2}})] \\ &+ \frac{1}{\sigma_2 t}\phi(\frac{\ln t - \mu_2}{\sigma_2})[1 - \Phi(\frac{\ln t(\frac{1}{\sigma_1} - \frac{\rho}{\sigma_2})}{\sqrt{1-\rho^2}} + \frac{\frac{\mu_2\rho}{\sigma_2} - \frac{\mu_1}{\sigma 1}}{\sqrt{1-\rho^2}})]. \end{aligned} \tag{4.3}$$

For the standard bivariate log-normal distribution, it reduces to

$$f_{T_1}(t) = \frac{2}{t}\phi(\ln t)[1 - \Phi(\ln t\sqrt{\frac{1-\rho}{1+\rho}})]. \tag{4.4}$$

Hence T_1 has a log-skew normal distribution. Using the result established earlier, the failure rate of T_1 is of the type U.

It can be verified that the turning point of $\eta_{T_1}(t)$ is given by the solution of the equation

$$\ln t + \lambda r_N(\lambda \ln t) = \lambda^2 r'_N(\lambda \ln t), \tag{4.5}$$

where $\lambda = \sqrt{(1-\rho)/(1+\rho}$ and $r_N(.)$ is the failure rate of the standard normal distribution.

For example, for $\lambda = \sqrt{\pi/2}$, the maximum value of $\eta_{T_1}(t)$ is achieved at $t = 1$.

Similarly, using (4.2), it can be verified that

$$f_{T_2}(t) = \frac{1}{\sigma_1 t}\phi(\frac{\ln t - \mu_1}{\sigma_1})[\Phi(\frac{\ln t(\frac{1}{\sigma_2} - \frac{\rho}{\sigma_1})}{\sqrt{1-\rho^2}} + \frac{\frac{\mu_1\rho}{\sigma_1} - \frac{\mu_2}{\sigma_2}}{\sqrt{1-\rho^2}}) + \frac{1}{\sigma_2 t}\phi(\frac{\ln t - \mu_2}{\sigma_2})[\Phi(\frac{\ln t(\frac{1}{\sigma_1} - \frac{\rho}{\sigma_2})}{\sqrt{1-\rho^2}} + \frac{\frac{\mu_2\rho}{\sigma_2} - \frac{\mu_1}{\sigma 1}}{\sqrt{1-\rho^2}})]. \quad (4.6)$$

For the standard bivariate log-normal distribution, the above expression reduces to

$$f_{T_2}(t) = \frac{2}{t}\phi(\ln t)[\Phi(\ln t\sqrt{\frac{1-\rho}{1+\rho}})]. \quad (4.7)$$

Again, T_2 has a log-skew normal distribution and its failure rate is of the type U.

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