# Random Effect Bivariate Survival Models and Stochastic Comparisons

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Abstract In this paper, we propose a general bivariate random effect model with special emphasis to frailty models and environmental effect models including some stochastic comparisons. The relationship between the conditional and the unconditional hazard gradients are derived and some examples are provided. We investigate as to how the well known stochastic orderings between the distributions of two frailties translate into the orderings between the corresponding survival functions. These results are used to obtain the properties of the bivariate multiplicative model and the shared frailty model.

## 1. INTRODUCTION

The random effect models are used in different deciplines. We shall present our results in the context of survival analysis or more specifically, in the context of frailty models where the frailty is modeled as an unobservable random effect. Clayton (1978) and Clayton and Cuzick (1985) introduced the proportional hazard frailty model, where a group of observations is assigned a random effect that acts multiplicatively on the baseline hazard function. The proportional hazard frailty model implies conditional independence- conditional on the frailty terms, the event times are independent. However, unconditionally, they are dependent.

**Frailty Models**

As is well known, a particular useful tool in handling heterogeneity unexplained by the observed covariates is the " frailty model" introduced by Vaupel et al. (1979). The classical frailty model is given by

$$\lambda(t|v) = v\lambda_0(t), t > 0, \tag{1.1}$$

where $\lambda_0(t)$ is the baseline hazard independent of $v$.

It is well known that the choice of frailty distribution strongly affects the estimate of the baseline hazard as well as the conditional probabilities, see Hougaard ( 1984, 1991, 1995, 2000), Heckman and Singer(1984) and Agresti et al. (2004). ). In this connection, Gupta and Kirmani (2006) investigated as to how well known stochastic orderings between distributions of two frailties translate into orderings between the corresponding survival functions. More recently. Gupta and Gupta (2009) studied similar problem for a general frailty model which includes the classical frailty model (1.1) as well as the additive frailty model.

For the model (1.1), the overall population hazard function $\lambda(t)$ is related to the baseline hazard function $\lambda_0(t)$ by the relation

$$\lambda(t) = \lambda_0(t)E(V|T > t)$$

Since

$$\frac{d}{dt}E(V|T>t) = -\lambda_0(t)Var(V|T>t),$$

see Gupta and Gupta (1996), $\lambda(t)/\lambda_0(t)$ is a decreasing function of $t$. It can be seen that if $E(V) \leq 1, \lambda(t) \leq \lambda_0(t), t > 0$, or equivalently $\overline{G}(t)/\overline{F}(t)$ is decreasing on $[0, \infty)$ where $\overline{G}(t)$ is the baseline survival function.

In this paper, we shall study a general bivariate frailty model and present some stochastic comparisons using different frailty distributions of the frailty. To do so, we define the bivariate hazard functions as follows:

Let $T_1$ and $T_2$ be two dependent random variables having absolutely continuous bivariate survival function $\overline{F}(t_1, t_2) = P(T_1 > t_1, T_2 > t_2)$. Then the hazard ( failure) rates $\lambda^{(i)}(t_1, t_2), i = 1, 2$ defined below are often used in demography, survival analysis and biostatistics when analyzing bivariate survival data. These are

$$\lambda^{(i)}(t_1, t_2) = -\frac{\partial}{\partial t_i} \ln \overline{F}(t_1, t_2), i = 1, 2. \tag{1.2}$$

Clearly $\lambda^{(1)}(t_1, t_2)$ is the hazard rate of $T_1$ given $T_2 > t_2$. Likewise $\lambda^{(2)}(t_1, t_2)$ is the hazard rate of $T_2$ given $T_1 > t_1$. The vector $(\lambda^{(1)}(t_1, t_2), \lambda^{(2)}(t_1, t_2)$ ) is called the hazard gadient. It is well knwon that the hazard gradient determines the survival function uniquely. Thus, we shall consider the general bivariate frailty model

$$\lambda^{(i)}(t_1, t_2|v) = \lambda^{(i)}(t_1, t_2, v), i = 1, 2, \tag{1.3}$$

where $v$ is the frailty associated with an individual. As mentioned earlier, our aim, in this paper, is to develop the properties of the general bivariate frailty model (1.3) and obtain some results for the stochastic comparisons using different frailty distributions. As a special case, we shall obtain results for the classical bivariate frailty model and the shared frailty model.

## 2. GENERAL BIVARIATE FRAILTY MODEL

Consider a general bivariate frailty model defined by the joint survival function $\overline{F}(t_1, t_2|v)$, of a two unit system, where $v$ is the frailty effect associated with the two variables. Define

$$\lambda^{(i)}(t_1, t_2|v) = \frac{f(t_i|T_j > t_j, v)}{\overline{F}(t_i|T_j > t_j, v)}, i \neq j, 1, 2.$$

That is $\lambda^{(i)}(t_1, t_2|v)$ is the failure rate function of the $ith$ unit with $jth(i \neq j)$ unit surviving at time $t_j$, conditional on the frailty varaiable $V$.

If $h(v)$ denotes the *pdf* of the random environmental effect $V$, then the unconditional joint survival function is

$$\overline{F}(t_1, t_2) = \int_0^\infty \overline{F}(t_1, t_2|v)h(v)dv,$$

The population level failure rate function is

$$\lambda^{(i)}(t_1,t_2) = \frac{f(t_i|T_j > t_j)}{\overline{F}(t_i|T_j > t_j)}, i \neq j, 1, 2.$$

In the following, we show that the population level hazard components are the averages of the conditional hazard components.

**Theorem 2.1.** *The population level failure rate function of the ith unit in a two unit system with jth unit of fixed age* $t_j$ *is the expected value of* $\lambda^{(i)}(t_1,t_2|v)$ *with respect to the conditional distribution of the frailty effect* $V$ *given* $T_1 > t_1$ *and* $T_2 > t_2$*. That is*

$$\lambda^{(i)}(t_1,t_2) = E_{V|T_1>t_1,T_2>t_2}[\lambda^{(i)}(t_1,t_2|v)], i = 1, 2.$$

Under a very mild condition, the following result addresses the monotonicity of the distribution (survival) function of the random effect as a function of the ages of the two units.

**Theorem 2.2.** *If* $\lambda^{(i)}(t_1,t_2|v)$ *is an increasing function of* $v, i = 1, 2,$ *then* $H(v|T_1 > t_1, T_2 > t_2)$ $(\overline{H}(v|T_1 > t_1, T_2 > t_2))$ *is an increasing (decreasing function of* $t_i$*, where*

$$H(v|T_1 > t_1, T_2 > t_2) = \frac{\int_0^v \overline{F}(t_i|T_j > t_j, u)h(u)du}{\overline{F}(t_i|T_j > t_j)}, i \neq j = 1, 2.$$

**Corollary 2.3.** *If* $\lambda^{(i)}(t_1,t_2|v)$ *is an increasing function of* $v, i = 1, 2,$ *then* $E(V|T_1 > t_1, T_2 > t_2)$ *is decreasing in* $t_i, i = 1, 2.$

**Remark 2.1.** *The statement in the above Corollary is a precise statement of the heuristically obvious fact that the weaker units in the population fail earlier than the others so that the remaining units are more robust than the rest.*

The following results compare the frailty distribution of two groups, one with $T_i > t_{i1}$ and $T_j > t_j$ and the other with $T_i > t_{i2}$ and $T_j > t_j$ , $t_{i1} < t_{i2}, i \neq j, i = 1, 2$.

**Theorem 2.4.** *If* $\lambda^{(i)}(t_1,t_2|v)$ *is an increasing function of* $v$*, then* $V|T_i > t_{i2}, T_j > t_j \leq_{LR} V|T_i > t_{i1}, T_j > t_j, 0 < t_{i1} < t_{i2}, i = 1, 2.$

The following result shows how the ordering between two frailties are preserved for surviving individuals.

**Theorem 2.5.** *Let* $V_1$ *and* $V_2$ *be two frailties random variables such that* $V_2 \leq_{LR} V_1$*. Then* $V_2|T_1 > t_1, T_2 > t_2 \leq_{LR} V_1|T_1 > t_1, T_2 > t_2.$

## 3. COMPARISONS OF FRAILTY MODELS

There is no firm basis for choosing the probability distribution of the frailty random variable $V$. It is , therefore, important to see how the overall survival function of the $ith$, unit, $i = 1, 2$ responds to the change in the probability distribution of $V$. Our main objective, in this section, is to see how some of the well known stochastic orderings between $V_1$ and $V_2$ translate into the orderings between the component lifetimes.

**Theorem 3.1.** *Let $\lambda^{(i)}(t_1, t_2|v)$ be an increasing function of $v > 0, i = 1, 2$.If $V_2 \leq_{LR} V_1$, then $(T_{11}, T_{21}) \leq_{WFR} (T_{12}, T_{22})$.*

The following result can be easily established

**Theorem 3.2.** *Suppose the conditional joint pdf of $T_i$ and $V$ given $T_j > t_j, i \neq j = 1, 2$ is $RR_2$. Then*

*(a) $V$ is stochastically decreasing in right tail with respect to $T_i$. That is $\overline{H}(v|T_i > t_i, T_j > t_j, i \neq j)$ is a decreasing function of $t_i, i = 1, 2$.*

*(b) $T_i$ is stochastically decreasing in right tail with respect to $V$ . That is $\overline{F}(t_i|V > v, T_j > t_j, i \neq j)$ is a decreasing function of $v$.*

The following result shows how the likelihood ratio ordering of $V_1$ and $V_2$ is inherited by $T_1$ and $T_2$.

**Theorem 3.3.** *Suppose $V_1 \leq_{LR} V_2$. If $f(t_i|v, T_j > t_j)$ is $RR_2$ $(TP_2)$ on $[0, \infty) \times [0, \infty)$, then*

$T_{i,v_1}|T_j > t_j \leq_{LR} T_{i,v_2}|T_j > t_j$.

The following result addresses the inheritence of failure rate orderings of $V_1$ and $V_2$ by $T_1$ and $T_2$.

**Theorem 3.4.** *Suppose (a) $V_1 \leq_{FR} V_2$ and*

*(b) $\lambda^{(i)}(t_1, t_2|v_1) \leq \lambda^{(i)}(t_1, t_2|v_2), v_1 < v_2$. Then*

$T_{i,v_1}|T_j > t_j \geq (\leq)_{FR}\ T_{i,v_2}|T_j > t_j, i, j = 1, 2, i \neq j$.

The following theorem shows the corresponding result for the stochastic orderings.

**Theorem 3.5.** *If $V_1 \leq_{st} V_2$ and $\overline{F}(t_i|v, T_j > t_j)$ is decreasing function of $v$, then $T_{i,v_2}|T_j > t_j \leq_{st} T_{i,v_1}|T_j > t_j, i, j = 1, 2, i \neq j$.*

## 4. MULTIPLICATIVE MODEL

We now consider the following model as a special case.

$$\lambda^{(i)}(t_1, t_2|v) = v\lambda_0^{(i)}(t_1, t_2), t_1 > 0, t_2 > 0, v > 0, \tag{4.1}$$

where $\lambda_0^{(i)}(t_1, t_2)$ is the baseline failure rate of the *ith* unit without taking the frailty effect and is independent of $v$.

We now present the following result:

**Theorem 4.1.** *For the model (4.1)*

*(a) The ith component of the population level failure rate is given by*

$$\lambda^{(i)}(t_1, t_2) = \lambda_0(t_1, t_2)E_V(V|T_1 > t_1, T_2 > t_2). \tag{4.2}$$

*(b) $H(v|T_1 > t_1, T_2 > t_2)$ is an increasing function of $t_i, i = 1, 2$.*

*(c) $E_V(V|T_1 > t_1, T_2 > t_2)$ is decreasing in $t_i > 0, i = 1, 2$. Moreover, if $\lambda_0^{(i)}(t_1, t_2)$ is decreasing in $t_i > 0$, then $\lambda^{(i)}(t_1, t_2)$ is decreasing in $t_i > 0$. That is $T_i|T_j > t_j, i \neq j = 1, 2$ is DFR.*

*(d) $V|T_i > t_{i2}, T_j > t_j \leq_{LR} V|T_i > t_{i1}, T_j > t_j$, for all $t_{i1} < t_{i2}, i \neq j = 1, 2$.*

*(e) If $V_2 \leq_{LR} V_1$, then $V_2|T_1 > t_1, T_2 > t_2 \leq_{LR} V_1|T_1 > t_1, T_2 > t_2$.*

**Theorem 4.2.** *$T_i$ is stochastically decreasing in the right tail with respect to $V$, given $T_j > t_j, i = 1, 2$. That is $\overline{F}(t_i|v, T_j > t_j)$ is a decreasing function of $v > 0$.*

**Theorem 4.3.** *(a) If $V_1 \leq_{LR} V_2$, then $T_{i,v_1}|T_j > t_j \geq_{LR} T_{i,v_2}|T_j > t_j, i \neq j = 1, 2$.*
*(b) If $V_1 \leq_{FR} V_2$, then $T_{i,v_1}|T_j > t_j \geq_{FR} T_{i,v_2}|T_j > t_j, i \neq j = 1, 2$.*
*(c) If $V_1 \leq_{st} V_2$, then $T_{i,v_1}|T_j > t_j \geq_{st} T_{i,v_2}|T_j > t_j, i \neq j = 1, 2$.*

## 5. Shared Frailty Model

We now consider the following model, known as the shared frailty model

$$\lambda^{(i)}(t_1, t_2|v) = v\lambda_{0i}(t_i), i = 1, 2, \tag{5.1}$$

where $\lambda_{0i}(t_i)$ is the baseline failure rate of the *ith* unit, independent of the other unit and of $v$.We now present the following result.

**Theorem 5.1.** *For the above model*
*(a) The population level ith component failure rate function is*

$$\lambda^{(i)}(t_1, t_2) = \lambda_{0i}(t_i)E_V(V|T_1 > t_1, T_2 > t_2).$$

*(b) $H(v|T_1 > t_1, T_2 > t_2)$ is an increasing function of $t_i, i = 1, 2$.*
*(c) $E_V(V|T_1 > t_1, T_2 > t_2)$ is decreasing in $t_i > 0, i = 1, 2$*
*Moreover, if $\lambda_{0i}(t_i)$ is decreasing in $t_i > 0$, then $\lambda^{(i)}(t_1, t_2)$ is decreasing in $t_i > 0$.*
*(d) $V|T_i > t_{i2}, T_j > t_j \leq_{LR} V|T_i > t_{i1}, T_j > t_j$ for all $t_{i1} < t_{i2}, i \neq j = 1, 2$.*
*(e) If $V_2 \leq_{LR} V_1$, then $V_2|T_1 > t_1, T_2 > t_2 \leq_{LR} V_1|T_1 > t_1, T_2 > t_2$.*

**Theorem 5.2.** *$T_i$ is (conditionally) stochastically decreasing in right tail with respect to $V, i = 1, 2$.That is ,$\overline{F}(t_i|v)$ is decreasing function of $v > 0, i = 1, 2$.*

Finally, we present the following result showing how the various stochastic orders between the frailties translate into the stochastic orderings between the failure times.

**Theorem 5.3.** *(a) If $V_1 \leq_{LR} V_2$, then $T_{i,v_1} \geq_{LR} T_{i,v_2}, i = 1, 2$*
*(b) IIf $V_1 \leq_{LR} V_2$, then $T_{i,v_1} \geq_{LR} T_{i,v_2}, i = 1, 2$*
*(c) If $V_1 \leq_{st} V_2$, then $T_{i,v_1} \geq_{st} T_{i,v_2}, i = 1, 2$*

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