A REVIEW OF TWEEDIE ASYMPTOTICS FOR LÉVY PROCESSES

Bent Jørgensen
Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55
5230 Odense M, Denmark
E-mail: bentj@stat.sdu.dk

Conventional large-sample asymptotics concerns limits of centered and scaled sums of i.i.d. random variables. More generally, we consider asymptotics for Lévy processes, which are real valued processes $X = \{X(t) : t \geq 0\}$ with independent stationary increments, satisfying $X(0) = 0$. Independent increments means that $X(t_i) - X(t_{i-1})$, $i = 1, \ldots, n$ are independent for all n and $0 \leq t_1 < t_2 < \cdots < t_n$, and stationary increments means that the distribution of $X(t+s) - X(t)$ is independent of t.

For motivation, let us consider a strictly α-stable Lévy process Z_α, such that $Z_\alpha(t)$ follows an α-stable distribution for some $\alpha \in (0, 2]$. This process satisfies the stability condition

$$(1) \qquad c^{-1/\alpha} Z_\alpha(ct) \stackrel{d}{=} Z_\alpha(t) \text{ for } t \geq 0$$

for all $c > 0$, where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions of the two processes. A Lévy process X is said to be in the domain of attraction of Z_α if

$$(2) \qquad c^{-1/\alpha} X(ct) \stackrel{d}{\rightarrow} Z_\alpha(t) \text{ for } t \geq 0$$

as $c \rightarrow \infty$, where $\stackrel{d}{\rightarrow}$ denotes convergence of the finite-dimensional distributions. This is the stable convergence theorem for Lévy processes, which is an example of large-sample asymptotics since it involves the asymptotic behaviour of $X(t)$ for t large.

We now add a drift $\mu \in R$ to both processes, defining

$$(3) \qquad X(\mu; t) = X(t) + \mu t \text{ and } Z_\alpha(\mu; t) = Z_\alpha(t) + \mu t.$$

We may then rewrite the stability condition (1) as follows:

$$(4) \qquad c^{-1/\alpha} Z_\alpha(\mu c^{1/\alpha - 1}; ct) \stackrel{d}{=} Z_\alpha(\mu; t) \text{ for } t \geq 0,$$

for all $c > 0$ and $\mu \in R$. Similarly, the convergence (2) may be rewritten as follows:

$$(5) \qquad c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \stackrel{d}{\rightarrow} Z_\alpha(\mu; t) \text{ for } t \geq 0,$$

as $c \rightarrow \infty$. We may interpret the left-hand side of (5) as a multiplicative transformation group $\{R_c^\alpha : c > 0\}$, defined by

$$(6) \qquad R_c^\alpha X(\mu; t) = c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct).$$

This is a continuous analogue of the renormalization group of Jona-Lasinio (1975). We say that (4) is a fixed point of the renormalization group (6), with domain of attraction consisting of Lévy processes with drift satisfying (5).

It seems plausible that the fixed point (4) is approached not just as c goes to infinity (large-sample asymptotics), but also as c goes to zero. In the latter case we consider the scaling limit of the process $X(\mu;t)$ for small t, and we hence refer to such results as *infinitely divisible asymptotics*. The following result, which is a special case of Theorem 6.1 of Jørgensen et al. (2011), confirms the possibility of infinitely divisible asymptotics.

Theorem 1 Consider a family of Lévy processes $Z(\mu;t) = Z(t) + \mu t$ with drift $\mu \in R$. If there exists a family of Lévy processes $X(\mu;t) = X(t) + \mu t$ with drift μ such that

(7) $$c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \xrightarrow{d} Z(\mu;t) \text{ for } t \geq 0 \text{ as } c \to \infty \text{ or } c \downarrow 0,$$

then $Z(\mu;t)$ is α-stable for each $\mu \in R$.

The large-sample version ($c \to \infty$) of Theorem 1 is a special case Lamperti's (1962) limit theorem for self-similar processes, and the proof in the infinitely divisible case ($c \downarrow 0$) is a straightforward modification of Lamperti's proof. The corresponding convergence (5) as $c \downarrow 0$ is equivalent to the limit (2) holding as $c \downarrow 0$. Not much is known about the domain of attraction to the fixed point (1) in the infinitely divisible limit, except that it contains the stable process Z_α, and hence is not empty. However, we shall now see that there exist natural exponential families of processes $X(\mu;\cdot)$ indexed by a real parameter μ which satisfy a convergence result similar to (7) without having the drift form (3).

Natural exponential families of Lévy processes

Let X denote a given Lévy process defined on a probability space $(\mathcal{X}, \mathcal{F}, P)$. Let κ denote the cumulant generating function of $X(1)$,

$$\kappa(\theta) = \log \mathbb{E}\left[e^{\theta X(1)}\right],$$

with effective domain $\Theta = \{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}$, and assume that $\Theta \neq \{0\}$. Following Küchler & Sørensen (1997, p. 8), for each $t \geq 0$ we define a class of probability measures $\left\{P_\theta^t : \theta \in \Theta\right\}$ on the space $(\mathcal{X}, \mathcal{F}_t)$ by

(8) $$\frac{dP_\theta^t}{dP^t} = \exp\left[\theta X(t) - t\kappa(\theta)\right],$$

where $\{\mathcal{F}_t : t \geq 0\}$ is a suitably defined right-continuous filtration and P^t denotes the marginal distribution of $X(t)$. This defines a unique probability measure P_θ on $(\mathcal{X}, \mathcal{F})$, such that P_θ^t is the restriction of P_θ to $\mathcal{F}_t$, and X is a Lévy process on the probability space $(\mathcal{X}, \mathcal{F}, P_\theta)$. The family $\{P_\theta : \theta \in \Theta\}$ is called the *natural exponential family of Lévy processes generated by* X, and we say that P_θ is obtained from P by *exponential tilting*.

For each $t > 0$, $\left\{P_\theta^t : \theta \in \Theta\right\}$ is a natural exponential family of probability measures on $\mathbb{R}$ with cumulant generating functions $s \mapsto t\left[\kappa(\theta + s) - \kappa(\theta)\right]$. The mean and variance of $X(t)$ are

$$\mathbb{E}_\theta\left[X(t)\right] = t\dot{\kappa}(\theta) = t\mu \text{ and } \mathrm{Var}_\theta\left[X(t)\right] = t\ddot{\kappa}(\theta) = tV(\mu),$$

say, where dots denote derivatives. Here the *rate parameter* $\mu = \dot{\kappa}(\theta)$ with domain $\overline{\Omega} = \dot{\kappa}(\Theta)$, defined by continuity at the boundary of Θ if necessary, is a one-to-one function of θ, and from now on we let $X(\mu;\cdot)$ denote the version of X corresponding to the probability measure $P_{\dot{\kappa}^{-1}(\mu)}$. Also, the function $V(\mu) = \ddot{\kappa} \circ \dot{\kappa}^{-1}(\mu)$ with domain $\Omega = \mathrm{int}\overline{\Omega}$ is the *variance function*, which characterizes $\left\{P_\theta^t : \theta \in \Theta\right\}$ among all natural exponential families, cf. Jørgensen (1997, p. 51). Consequently, $\{P_\theta : \theta \in \Theta\}$ is also characterized by V among all natural exponential families of Lévy processes.

Tweedie asymptotics for Lévy processes

Following Lee & Whitmore (1993), we define the Hougaard Lévy processes $S_p(\mu;\cdot)$ for each p in $\Delta = \mathbb{R} \setminus (0,1)$ to be the natural exponential families of Lévy processes corresponding to the power variance functions $V(\mu) = \mu^p$. Here the domain for μ is $\Omega_0 = \mathbb{R}$ for $p = 0$ and $\Omega_p = \mathbb{R}_+$ for $p \in \Delta \setminus \{0\}$. We define the parameter $\alpha \in (-\infty, 1) \cup (1, 2]$ by

(9) $$\alpha = \alpha(p) = 1 + (1-p)^{-1} \text{ for } p \in \Delta$$

with the convention that $\alpha(1) = -\infty$. We shall now see that there is a close parallel between the Hougaard Lévy process $S_p(\mu;\cdot)$ and the stable process with drift $Z_\alpha(\mu;\cdot)$. In the following we use α and p freely in the notation.

The Hougaard Lévy process $S_p(\mu;\cdot)$ is a fixed point of the renormalization group (6) for $p \neq 2$ ($\alpha \neq 0$), satisfying

(10) $$c^{-1/\alpha} S_p(\mu c^{1/\alpha - 1}; ct) \stackrel{d}{=} S_p(\mu; t) \text{ for } t \geq 0$$

for all $\mu \in \Omega_p$ and $c > 0$. This follows by observing that for given $c > 0$, each side of (10) is a natural exponential family of Lévy processes with variance function $V(\mu) = \mu^p$. The corresponding domain of attraction consists of families of Lévy processes $X(\mu;\cdot)$ satisfying

(11) $$c^{-1/\alpha} X(\mu c^{1/\alpha - 1}; ct) \xrightarrow{d} S_p(\mu; t) \text{ for } t \geq 0 \text{ as } c \downarrow 0 \text{ or } c \to \infty$$

for $\mu \in \Omega_p$, corresponding to infinitely divisible and large-sample asymptotics, respectively. By the Tweedie convergence theorem of Jørgensen et al. (1994), an important part of the domain of attraction may be characterized in terms of the power asymptotic behaviour of the variance function V for X,

(12) $$V(\mu) \sim \mu^p \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty,$$

see also Jørgensen (1997, p. 148). Which asymptote is required depends on whether the power $c^{1/\alpha - 1}$ tends to zero or infinity in (11), see the examples below. More generally, Jørgensen et al. (2009) characterized the domain of attraction (11) corresponding to regular variation of V at zero or infinity.

Examples of Tweedie convergence

We now consider the main special cases of Tweedie asymptotics for $p \in \Delta$. In the following we let $\{X(\mu;\cdot) : \mu \in \overline{\Omega}\}$ denote a natural exponential family of Lévy processes with variance function V.

Brownian motion with drift ($p = 0$, or $\alpha = 2$): In this case $S_0(\mu;\cdot)$ is Brownian motion with drift μ, (a 2-stable Lévy process), i.e.

$$S_0(\mu; t) = B(t) + \mu t,$$

where $B(t)$ is ordinary Brownian motion. This is the only Hougaard Lévy process where exponential tilting corresponds to a drift term (the Cameron-Martin-Girsanov formula). The process $S_0(\mu;\cdot)$ satisfies the fixed point

$$c^{-1/2} S_0(\mu c^{1/2 - 1}; ct) \stackrel{d}{=} S_0(\mu; t) \text{ for } t \geq 0$$

for all $\mu \in \mathbb{R}$ and $c > 0$. The case of large-sample asymptotics is obtained for $V(\mu) \sim 1$ as $\mu \downarrow 0$, for which

(13) $$c^{-1/2} X(\mu c^{-1/2}; ct) \xrightarrow{d} S_0(\mu; t)$$

as $c \to \infty$. The case of infinitely divisible asymptotics is obtained for $V(\mu) \sim 1$ as $\mu \to \infty$, where (13) holds as $c \downarrow 0$. We note here that the convergence (13) is different from the corresponding version of (5) with $\alpha = 2$, because the exponential tilting of $X(\mu; \cdot)$ does not in general correspond to a drift term.

Poisson process ($p = 1$, or $\alpha = -\infty$): The Poisson process $S_1(\mu; \cdot)$ with rate $\mu > 0$ satisfies the fixed point

$$S_1(\mu c^{-1}; ct) \stackrel{d}{=} S_1(\mu; t) \text{ for } t \geq 0,$$

which is in the well-known scaling property of the Poisson process. The case of large-sample asymptotics is obtained for $V(\mu) \sim \mu$ as $\mu \downarrow 0$, which holds if the distribution of $X(\mu; 1)$ has positive probability mass at 0, and $(0, 1)$ is the largest interval starting at zero with zero probability, see Jørgensen et al. (1994). We then have convergence to the Poisson process as follows:

(14) $$X(\mu c^{-1}; ct) \stackrel{d}{\to} S_1(\mu; t) \text{ for } t \geq 0,$$

as $c \to \infty$ for all $\mu > 0$. The case of infinitely divisible asymptotics is obtained for $V(\mu) \sim \mu$ as $\mu \to \infty$, where (14) holds as $c \downarrow 0$.

Compound Poisson processes ($1 < p < 2$, or $\alpha < 0$): This class of Hougaard Lévy processes corresponds to gamma compound Poisson processes with gamma jumps, see Jørgensen (1997,Ch. 4). For given $\mu > 0$, let us write $\mu = \mu_1\mu_2$, where

$$\mu_1 = \frac{\alpha - 1}{\alpha}\mu^{\alpha/(\alpha-1)} \text{ and } \mu_2 = \frac{\alpha}{\alpha - 1}\mu^{1/(1-\alpha)}.$$

The process $S_p(\mu; \cdot)$ is then defined as follows:

(15) $$S_p(\mu; t) = \sum_{i=1}^{S_1(\mu_1;t)} X_i,$$

where the X_i are i.i.d. gamma variables with mean μ_2 and shape parameter $-\alpha > 0$, independent of the Poisson process $S_1(\mu_1; t)$. The process $S_p(\mu; \cdot)$ satisfies the fixed point (10) with domain of attraction including the power asymptotic behaviour of V (12). In the special case $p = 3/2$ ($\alpha = -1$) we obtain the compound Poisson process with exponential jumps, which satisfies the fixed point

$$cS_{3/2}(\mu c^{-2}; ct) \stackrel{d}{=} S_{3/2}(\mu; t) \text{ for } t \geq 0$$

for all $\mu, c > 0$. If X is in the domain of attraction of $S_{3/2}(\mu; \cdot)$ we obtain

$$cX(\mu c^{-2}; ct) \stackrel{d}{\to} S_{3/2}(\mu; t) \text{ for } t \geq 0$$

as $c \downarrow 0$ or $c \to \infty$. The case of infinitely divisible asymptotics is obtained for $V(\mu) \sim \mu^{3/2}$ as $\mu \to \infty$. Large-sample asymptotics is obtained for $V(\mu) \sim \mu^{3/2}$ as $\mu \downarrow 0$, which holds for example if the distribution of $X(\mu; 1)$ is continuous on $\mathbb{R}_+$ with a positive probability mass at 0, see Jørgensen et al. (1994).

Gamma process ($p = 2$, or $\alpha = 0$): The gamma Lévy process $S_2(\mu; \cdot)$ with rate $\mu > 0$ satisfies the following scaling property for each $\mu, c > 0$,

(16) $$c^{-1}S_2(\mu c; t) \stackrel{d}{=} S_2(\mu; t) \text{ for } t > 0,$$

which is like a fixed point, but not of the form (10). The corresponding domain of attraction consists of natural exponential families of Lévy processes $X(\mu;\cdot)$ satisfying

$$c^{-1}X(\mu c;t) \xrightarrow{d} S_2(\mu;t) \text{ for } t \geq 0 \text{ as } c \downarrow 0 \text{ or } c \to \infty,$$

which is obtained if V satisfies

$$V(\mu) \sim \mu^2 \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty,$$

respectively. This type of convergence was further characterized by Jørgensen et al. (2009) in terms of regular variation of V.

α-stable processes ($p > 2$ or $p < 0$, corresponding to $0 < \alpha < 1$ or $1 < \alpha < 2$): The process $S_p(\mu;\cdot)$ satisfies the fixed point (10), which in particular implies that $S_p(\infty;\cdot)$ is a positive α-stable Lévy process for $0 < \alpha < 1$ ($p > 2$), whereas $S_p(0;\cdot)$ is an extreme α-stable Lévy process for $1 < \alpha < 2$ ($p < 0$). In these cases the Hougaard Lévy processes $S_p(\mu;\cdot)$ are hence exponential tiltings of either the process $S_p(\infty;\cdot)$ or $S_p(0;\cdot)$, respectively.

The process $S_3(\mu;\cdot)$ is an inverse Gaussian process with rate $\mu > 0$, which is an exponential tilting of the 1/2-stable process $S_3(\mu;\cdot)$, see also Wasan (1968) or Chhikara & Folks (1989, p. 193). The Inverse Gaussian process satisfies the fixed point

$$c^{-2}S_3(\mu c;ct) \stackrel{d}{=} S_3(\mu;t) \text{ for } t \geq 0,$$

with corresponding domain of attraction

$$c^{-2}X(\mu c;ct) \xrightarrow{d} S_3(\mu;t) \text{ for } t \geq 0, \tag{17}$$

for all $\mu, c > 0$. The power asymptotic behaviour

$$V(\mu) \sim \mu^3 \text{ as } \mu \downarrow 0 \text{ or } \mu \to \infty$$

implies the infinitely divisible ($c \downarrow 0$) or large-sample ($c \to \infty$) convergence for (17), respectively.

For $p < 0$ ($1 < \alpha < 2$) the power asymptotics $V(\mu) \sim \mu^p$ as $\mu \downarrow 0$ or $\mu \to \infty$ implies the large-sample ($c \to \infty$) or infinitely divisible ($c \downarrow 0$) convergence in (11), respectively.

REFERENCES

Chhikara, R.S., & Folks, J.L. (1989). The Inverse Gaussian Distribution: Theory, Methodology, and Applications. *New York: Marcel Dekker.*

Jona-Lasinio, G. (1975). The renormalization group: a probabilistic view. *Il Nuovo Cimento, 26, 99–119.*

Jørgensen, B. (1997). The Theory of Dispersion Models. *London: Chapman & Hall.*

Jørgensen, B., Demétrio, C.G.B., & Martínez, J.R. (2011). Self-similarity and Lamperti convergence for families of stochastic processes. *Lithuanian Math. J. (to appear).*

Jørgensen, B., Martínez, J.R., & Tsao, M. (1994). Asymptotic behaviour of the variance function. *Scand. J. Statist., 21, 223–243.*

Jørgensen, B., Martínez, J.R., & Vinogradov, V. (2009). Domains of attraction to Tweedie distributions. *Lithuanian Math. J., 49, 399–425.*

Küchler, U., & Sørensen, M. (1997). Exponential Families of Stochastic Processes. *New York: Springer-Verlag.*

Lamperti, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc. 104, 62–78.*
Lee, M.-L.T., & Whitmore, G.A. (1993). Stochastic processes directed by randomized time. *J. Appl. Probab., 30, 302–314.*
Wasan, M.T. (1968). On an inverse Gaussian process. *Skand. Aktuarietidskr., 69–96.*

RÉSUMÉ

Conventional central limit theory concerns centered and scaled sums of i.i.d. random variables with finite variances. The corresponding enormous domain of attraction of the normal distribution is in a way also its biggest liability, because the normal approximation is based on first and second moments only. We review the topic of Tweedie asymptotics for natural exponential families of Lévy processes, which involves exponentially tilted and scaled sums of i.i.d. random variables. Here the three-parameter family of Tweedie exponential dispersion models appears in the limit, including normal, Poisson, compound Poisson, gamma, and exponentially tilted positive and extreme stable distributions.