# On cycle representations of random walks in fixed, random environments

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**1. Introduction.** A systematic work has been developed (S. Kalpazidou [1], S. Kalpazidou and Ch. Ganatsiou [2], J. MacQueen [3], Qian Minping and Qian Min [4], A.H. Zemanian [5] and others) in order to investigate representations of the distributions of Markov processes (with discrete or continuous parameter) having an invariant measure as decompositions in terms of the *cycle (or circuit) passage functions*. The representations are called *cycle (or circuit) representations* while the corresponding discrete parameter Markov processes generated by directed cycles (or circuits) are called *cycle (or circuit) chains.*

Following the context of the theory of *Markov processes' cycle-circuit representation* (S. Kalpazidou [1]), the present work arises as an attempt to investigate a proper unique representation by directed cycles (especially by directed circuits) and weights of a random walk with jumps (having absorbing barriers) in a fixed, random environment. For such random walks in ergodic fixed, random environments these representations will give us the possibility in future research to study the proper criterions regarding transience or recurrence of the corresponding Markov chains.

**2. Auxiliary results**. Let S be a denumerable set. The directed sequence $c = (i_1, i_2, \ldots, i_m, i_1)$ modulo the cyclic permutations, where $i_1, i_2, \ldots, i_m \in S$, $m>1$, completely defines a *directed circuit* in S. The ordered sequence $\hat{c} = (i_1, i_2, \ldots, i_m)$ associated with the given directed circuit c is called *a directed cycle* in S.

A directed circuit may be considered as $c = (c(m), c(m+1), \ldots, c(m+v-1), c(m+v))$, if there exists an $m \in \mathbb{Z}$, such that $i_1 = c(m+0)$, $i_2 = c(m+1)$, …, $i_v = c(m+v-1)$, $i_1 = c(m+v)$, that is a periodic function from $\mathbb{Z}$, the set of integers, to S.

The smallest integer $p \equiv p(c) \geq 1$ satisfying the equation $c(m+p) = c(m)$, for all $m \in \mathbb{Z}$, is the *period* of c. A directed circuit c such that p(c)=1 is called a *loop*. (In the present work we shall use directed circuits with distinct point elements).

Let a directed circuit c (or a directed cycle ĉ) with period p(c) > 1. Then we may define by

$$J_c^{(n)}(i,j) = \begin{cases} 1, & \text{if there exists an } m \in Z \text{ such that } i = c(m),\ j = c(m+n), \\ 0, & \text{otherwise} \end{cases}$$

the *n-step passage function* associated with the directed circuit c, for any $i, j \in S$, $n \geq 1$.
For $n = 1$ we have

$$J_c^{(1)}(i,j) = \begin{cases} 1, & \text{if there exists an } m \in Z \text{ such that } i = c(m),\ j = c(m+1), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore we may define by

$$J_c(i) = \begin{cases} 1, & \text{if there exists an } m \in Z \text{ such that } i = c(m), \\ 0, & \text{otherwise} \end{cases}$$

the *passage function* associated with the directed circuit c, for any $i \in S$.

Given a denumerable set S and an infinite denumerable class C of overlapping directed circuits (or directed cycles) with distinct points (except for the terminals) in S such that all the points of S can be reached from one another following paths of circuit-edges, that is, for each two distinct points i and j of S there exists a sequence $c_1, ..., c_k$, $k \geq 1$, of circuits (or cycles) of C such that i lies on $c_1$ and j lies on $c_k$ and any pair of consecutive circuits $(c_n, c_{n+1})$ have at least one point in common.

With each directed circuit (or directed cycle) $c \in C$ let us associate a strictly *positive weight* $w_c$ which must be independent of the choice of the representative of c, that is, it must satisfy the consistency condition $w_{c \circ t_k} = w_c$, $k \in \mathbb{Z}$, where $t_k$ is the translation of length k (that is, $t_k(n) \equiv n+k$, $n \in \mathbb{Z}$, for any fixed $k \in \mathbb{Z}$).

For a given class C of overlapping directed circuits (or cycles) and for a given sequence $(w_c)_{c \in C}$ of weights we may define by

$$p_{ij} = \frac{\sum_{c \in C} w_c \cdot J_c^{(1)}(i, j)}{\sum_{c \in C} w_c \cdot J_c(i)},$$

the elements of a Markov transition matrix on S, iff $\sum_{c \in C} w_c \cdot J_c(i) < \infty$, for any $i \in S$. This means that a given Markov transition matrix $P = (p_{ij})$, $i, j \in S$, can be represented by directed circuits (or cycles) and weights iff there exists a class of overlapping directed circuits (or cycles) C and a sequence of positive weights $(w_c)_{c \in C}$ such that the abovementioned formula holds. In this case the Markov transition matrix P has an invariant measure p which is a solution of pP=p defined by

$$p(i) = \sum_{c \in C} w_c \cdot J_c(i), \quad i \in S.$$

The following classes of Markov chains may be represented uniquely by circuits (or cycles) and weights:

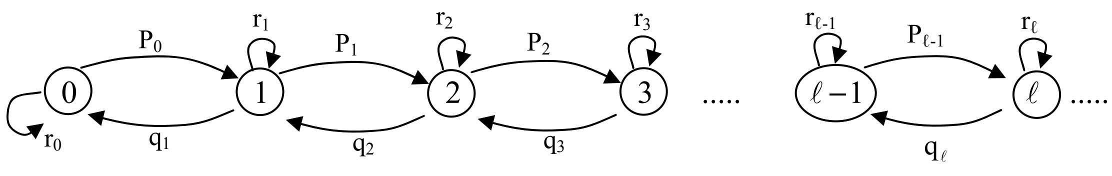
*(i) the recurrent Markov chains* (Qian Minping and Qian Min [4]).

*(ii) the reversible Markov chains.*

**3. Markov chains' cycle representation of a random walk with jumps on $\mathbb{N}$ in a fixed environment.** Let us consider a Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{N}$ with transitions $k \to (k+1)$, $k \to (k-1)$ and $k \to k$ whose the elements of the corresponding Markov transition matrix are defined as

$$\begin{aligned}
&P(X_{n+1} = 0 \,/\, X_n = 0) = r_0 \\
&P(X_{n+1} = 1 \,/\, X_n = 0) = p_0, && p_0 = 1 - r_0, \\
&P(X_{n+1} = k+1 \,/\, X_n = k) = p_k, && k \geq 1, \\
&P(X_{n+1} = k \,/\, X_n = k) = r_k, && k \geq 1, \\
&P(X_{n+1} = k-1 \,/\, X_n = k) = q_k, && k \geq 1,
\end{aligned}$$

such that $p_k + q_k + r_k = 1$, $0 \leq p_k$, $r_k \leq 1$, for every $k \geq 1$, as it is shown in the following diagram:




Assume that $(p_k)_{k \geq 0}, (r_k)_{k \geq 0}$ are arbitrary fixed sequences with $0 \leq p_0 = 1 - r_0 \leq 1$, $0 \leq p_k$, $r_k \geq 1$, for every $k \geq 1$.

Consider the directed circuits $c_k = (k, k+1, k)$, $c'_k = (k, k)$, $k \geq 0$ and the collections of weights $(w_{c_k})_{k\geq 0}$ and $(w_{c'_k})_{k\geq 0}$ respectively. We may obtain that

$$p_k = \frac{w_{c_k}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}}, \text{ for every } k \geq 1, \text{ with}$$

$$p_0 = \frac{w_{c_0}}{w_{c_0} + w_{c'_0}}.$$

Here the class C(k) contains the directed circuits $c_k = (k, k+1, k)$, $c_{k-1} = (k-1, k, k-1)$, $c'_k = (k, k)$. Furthermore we may define

$$q_k = \frac{w_{c_{k-1}}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}}, \text{ for every } k \geq 1$$

and

$$r_k = \frac{w_{c'_k}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}}, \text{ such that } p_k + q_k + r_k = 1, \text{ for every } k \geq 1, \text{ with}$$

$$r_0 = 1 - p_0 = \frac{w_{c'_0}}{w_{c_0} + w_{c'_0}}.$$

The transition matrix $P = (p_{ij})$ with

$$p_{ij} = \frac{\sum_{k=0}^{\infty} w_{c_k} \cdot J^{(1)}_{c_k}(i, j)}{\sum_{k=0}^{\infty} \left[w_{c_k} \cdot J_{c_k}(i) + w_{c'_k} \cdot J_{c'_k}(i)\right]}, \text{ for } i \neq j \text{ and } p_{ii} = \frac{\sum_{k=0}^{\infty} w_{c'_k} \cdot J_{c'_k}(i)}{\sum_{k=0}^{\infty} \left[w_{c_k} \cdot J_{c_k}(i) + w_{c'_k} \cdot J_{c'_k}(i)\right]},$$

where $J^{(1)}_{c_k}(i, j) = 1$, if $i, j$ are consecutive points of the circuit $c_k$,

$J_{c_k}(i) = 1$, if $i$ is a point of the circuit $c_k$,

$J_{c'_k}(i) = 1$, if $i$ is a point of the circuit $c'_k$,

expresses the representation of the Markov chain $(X_n)_{n\geq 0}$ by cycles (especially by directed circuits) and weights.

So we have the following

**Proposition 1.** *The Markov chain $(X_n)_{n\geq 0}$ describes a random walk as above has a unique representation by cycles (especially by directed circuits) and weights.*

**Proof.** Les us consider the set of directed circuits $c_k = (k, k+1, k)$ and $c'_k = (k, k)$, for every $k \geq 0$, since only the transitions from k to k+1, k to k-1 and k to k are possible. There are three circuits through each point $k \geq 1$: $c_{k-1}$, $c_k$, $c'_k$ and two circuits through 0 $c_0$, $c'_0$.

The problem we have to manage is the definition of the weights. We may symbolize by $w_k$ the weight $w_{c_k}$ of the circuit $c_k$ and by $w'_k$ the weight $w_{c'_k}$ of the circuit $c'_k$, for every $k \geq 0$. The sequences $(w_k)_{k\geq 0}$, $(w'_k)_{k\geq 0}$ must be solutions of

$$p_k = \frac{w_k}{w_{k-1} + w_k + w'_k}, \quad k \geq 1 \quad \text{with} \quad p_0 = \frac{w_0}{w_0 + w'_0},$$

$$r_k = \frac{w'_k}{w_{k-1} + w_k + w'_k}, \quad k \geq 1 \quad \text{with} \quad r_0 = \frac{w'_0}{w_0 + w'_0},$$

$$q_k = 1 - p_k - r_k, \quad k \geq 1.$$

Consider the sequences $(b_k)_k$, $(\gamma_k)_k$ where $b_k = \frac{w_k}{w_{k-1}}$, $\gamma_k = \frac{w'_k}{w'_{k-1}}$, $k \geq 1$.

As a consequence we may have

$$b_k = \frac{p_k}{1 - p_k - r_k}, \quad \gamma_k = \frac{r_k}{r_{k-1}} \cdot \frac{P_{k-1}}{P_k} \cdot b_k, \quad \text{for every } k \geq 1.$$

For given sequences $(p_k)_{k\geq 0}$, $(r_k)_{k\geq 0}$ it is clear that $(b_k)_{k\geq 1}$, $(\gamma_k)_{k\geq 1}$ exist and are unique for those sequences. This means that the sequences $(w_k)_{k\geq 0}$, $(w'_k)_{k\geq 0}$ are defined uniquely, up to multiplicative constant factors, by

$$w_k = w_0 \,.\, b_1 \ldots b_k,$$

$$w'_k = w'_0 \,.\, \gamma_1 \ldots \gamma_k \,.$$

(the unicity is based to the constant factors $w_0$, $w'_0$).

**4. Markov chains' cycle representation of a random walk with jumps in a random environment.** Let us now consider a random walk on ℤ with only possible jumps (-1), (+1) or into itself and with transition probabilities $(p_k)_{k\in Z}$, $(q_k)_{k\in Z}$, $(r_k)_{k\in Z}$ (where $q_k = 1 - p_k - r_k$, $k \in Z$) which describe stationary ergodic sequences.

Let also $(\Omega, \mathfrak{F}, \mu)$ be a probability space, u a measure preserving ergodic automorphism of this space and $p : \Omega \to (0, 1)$, $r : \Omega \to (0, 1)$ measurable functions. Then every $\omega \in \Omega$ creates the random environments $p_k = p(u^k\omega)$, $r_k = r(u^k\omega)$.
Since u is measure preserving and ergodic, the sequences $(p_k)_k$, $(r_k)_k$ are stationary, ergodic sequences of random variables. This gives us the possibility to define on the infinite product space $\mathbb{Z}^{\mathbb{N}}$ with coordinates $(X_n)_n$ a family $(P^\omega)_{\omega\in\Omega}$ of probability measures such that for every $\omega \in \Omega$

$$P^\omega (X_{n+1} = k+1 \,/\, X_n = k) = p\,(u^k\omega),$$
$$P^\omega (X_{n+1} = k \,/\, X_n = k) = r\,(u^k\omega),$$
$$P^\omega (X_{n+1} = k-1 \,/\, X_n = k) = 1-p\,(u^k\omega) - r(u^k\omega) = q(u^k\omega).$$

Under this construction $(X_n)_n$ is a Markov chain on ℤ with transition probabilities described as above. By using an analogous proof of that given before and by taking the sequences $(b_k(\omega))_k$, $(\gamma_k(\omega))_k$, where $b_k(\omega) = \dfrac{w_k(\omega)}{w_{k-1}(\omega)}$, $\gamma_k(\omega) = \dfrac{w'_k(\omega)}{w'_{k-1}(\omega)}$, $k\in\mathbb{Z}$, we may obtain that, for $\mu$ almost every environment $\omega \in \Omega$, the Markov chain $(X_n)_n$ has a representation by cycles (especially by directed circuits) and weights.

## REFERENCES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