Simulation of Tail Dependence in Cot-copula

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Abstract

Tail dependence copulas provide an efficacious tool to capture tail dependence of a multivariate distribution. Two popular copulas, Gumble and Claytron, capture upper and lower tail dependence respectively. In this paper, we propose a new bivariate copula, namely the Cot-copula which capture both upper and lower tail dependence and these measures coincide with that of Gumbel and Clayton tail dependence measures, respectively. The propose copula also has a wider dependence coverage for the Kendall's tau (τ) than the 12^{th} family of Archimedean Copula of (Nelsen, 2006), which illustrates the ability to capture a wider range of dependence structure.

Keywords: Archimedean Copulas, Cot-copula, Tail dependence, Kendall's tau, Dependence Coverage.

Introduction

A copula is a function which binds or 'couples' an $n-$ dimensional distribution to its one-dimensional margins and is itself a continuous distribution function which characterizes the dependence structure of the model. This is indeed useful to risk management(Bouyé, Durrleman, Nikeghbali, Riboulet, & Roncalli, 2001).

The statistical analysis of the distribution of individual asset returns frequently finds fat tails, skewness, and other non-normal features which leads to underestimation of this dependence measure (see for example (Ang & Bekaert, 2002; Ang & Chen, 2002; Bae, Karolyi, & Stulz, 2003; Longin & Solnik, 2001). This has led many to consider other alternatives and the introduction of copulas as flexible methods of multivariate modeling is very timely.

The Archimedean copulas are an important family of copulas, which have a simple form with properties such as associativity and have a variety of dependence structures. (C. Genest & J. MacKay, 1986; C. Genest & R. Mackay, 1986; Joe, 1997; Müller & Scarsini, 2005; Nelsen, 2006). Some important applications of the Archimedean copulas can be found in the studies of marketing, finance for example, (Coutant, Martineu, Messines, Riboulet, & Roncalli, 2001) , and rainfall (AghaKouchak, Bárdossy, & Habib, 2010).

In order to characterize the dependence of extreme risk, the concept of tail dependence for bivariate distribution functions was introduced by (Joe, 1997). With the exception of 12^{th} family, most Archimedean copulas introduced in Table 4.1 of Nelsen (Nelsen, 2006) cannot explain both tail behaviour observed on financial markets(Nelsen, 2005) . In order to obtain copulas with bivariate tail dependence measures, many authors construct new-copulas as a convex linear combination of two copulas; examples are Joe-Clayton (Joe, 1997), Gumbel-Clayton (Ane & Kharoubi, 2003) and many more. The aim of this paper is to introduce a new bivariate Archimedean copula with one-parameter family of generator C_θ, namely Cot-copula, which has bivariate tail dependences, and are comparable to those of the established Gumbel and Clayton copulas. In addition, the proposed copula is able to capture a wider range of dependence structure since it has wider dependence coverage for the Kendall's τ than the 12th family of Archimedean Copula of (Nelsen, 2006).

This paper is organized as follows. In Section 2, we provide preliminaries for the Archimedean copulas and introduce the Cot-Copula. Section 3 derives the tail dependence measure and Kendall's tau measure of the proposed copula. This is followed by "Empirical application" sections and "*Conclusion*" which contains some concluding remarks.

Preliminaries: The Copula

A copula C is a distribution function of a random vector in $\Re^2$, each with a uniform marginal distribution, with the following properties as given in Theorem 1.1.

<u>Theorem 1.1:</u> A function $C:[0,1]^2 \to [0,1]$ is a copula iff the following properties hold:

i. $C(u_1,u_2)=0$ *for* $u_1=0$ *or* $u_2=0$,

ii. $C(u_1,1)=u_1, C(u_2,1)=u_2$ *for all* u_1 *and* u_2 *in the unit interval [0,1]*,

iii. $\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\, C\left(u_{1i},u_{2,j}\right)\geq 0$ *for all* $\left(u_{1,i},u_{2,j}\right)$ *in* $[0,1]$ *with* $u_{1,1}<u_{1,2}$ *and* $u_{2,1}<u_{2,2}$.

Thus, for joint distribution function H with margins $F_1(X_1), F_2(X_2)$ there is a copula C such that equation $H(x_1,x_2)=C(F_1(X_1),F_2(X_2))$. holds. This copula, C, is unique if the marginal distributions are continuous.

Cot- Copula

A bivariate Archimedean copula C can be generated by considering a class Φ of functions $\varphi:(0,1]\to[0,\infty)$ which are continuous, strictly decreasing, convex, and for which $\phi(1)=0$. This copula based on its generator φ can be constructed by following formula:

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v)), \qquad 0\leq u,v\leq 1, \tag{1}$$

Where $\varphi^{[-1]}$ is the pseudo-inverse of continues and strictly decreasing function φ with Dom $\varphi^{[-1]}=[0,\infty)$, Rand $\varphi^{[-1]}=[0,1]$ and

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t) & 0\leq t\leq\varphi(0),\\ 0 & \varphi(0)\leq t\leq\infty.\end{cases} \tag{2}$$

An important subclass of Φ, as noted by (Nelsen, 2006), consists of those elements of φ which has two continuous derivatives with $\varphi'(t)<0$ and $\varphi''(t)>0$ for $t\in(0,1)$. As an extension to the Archimedean family, we propose a new generator defined as:

$$\varphi(t)=\cot^{\theta}(\frac{\pi}{2}t) \qquad \theta\geq 1. \tag{3}$$

The condition $\theta\geq 1$ in equation (3) guarantees the following properties of this generator function $\varphi(t)$:

$$\varphi(1)=\cot^{\theta}(\frac{\pi}{2})=0$$

$$if\,\theta \geq 1 \rightarrow \varphi'(t) = (-\theta\frac{\pi}{2})\cot^{\theta-1}(\frac{\pi}{2}t)(1+\cot^2(\frac{\pi}{2}t)) < 0,$$

$$if\,\theta \geq 1 \rightarrow \varphi''(t) = (\theta\frac{\pi}{2})(\frac{\pi}{2}\cot^{\theta-2}(\frac{\pi}{2}t)(1+\cot^2(\frac{\pi}{2}t))((\theta-1)+(\theta+1)\cot^2(\frac{\pi}{2}t)) > 0.$$

In addition, $\varphi(0) = \lim_{t\to 0}\cot^{\theta}(\frac{\pi}{2}t) = \infty$ suffices to guarantee that the strict inverse exists, that is:

$$\varphi^{[-1]}(t) = \varphi^{-1}(t) = \frac{2}{\pi}arc\cot(t^{\frac{1}{\theta}}). \tag{4}$$

From (1), the corresponding copula is then defined by the following function.

$$C(u,v) = \frac{2}{\pi}arc\cot(\cot^{\theta}(\frac{\pi}{2}u)+\cot^{\theta}(\frac{\pi}{2}v))^{\frac{1}{\theta}}, \qquad \theta \geq 1. \tag{5}$$

Dependence Measure

The role of copula in dependence can be considered in two ways. First, it describes the dependence structure as a consequence of Sklar’s theorem. Secondly, since copula is invariant under strictly increasing transformation, this provides a way of studying scale invariant measure of association. Here we consider two such measures for the Cot-copula: tail dependence measures which characterize the dependences between extreme values which is highly important in finance, and Kendall’s τ measure of association. For details of these measures, see (Abdi, 2007; Ane & Kharoubi, 2003; Joe, 1997; Kendall, 1938; D. Li; D. X. Li, 2000; Nelsen, 2006).

One of the most important statistical properties of copula is dependence coverage, that is, the range of dependence structure that a copula can capture. The usefulness of copula family in modeling can often depend on its dependence coverage. Based on Kendall’s τ we can show that the proposed Archimedean copula has rather wider dependence coverage in compare with 12^{th} family of Archimedean copula.

Tail Dependence Measures

When C is Archimedean with generator φ, the upper tail dependence can be expressed as:

$$\lambda_u = 2 - \lim_{t\to 0^+}\frac{1-\varphi^{[-1]}(2t)}{1-\varphi^{[-1]}(t)}. \tag{6}$$

Similarly, the lower tail dependence parameter λ_l is

$$\lambda_l = \lim_{t\to\infty}\frac{\varphi^{[-1]}(2t)}{\varphi^{[-1]}(t)}. \tag{7}$$

For the proposed generator $\varphi(t) = \cot^{\theta}(\frac{\pi}{2}t)$, the upper and lower tail dependence (λ_u and λ_l respectively) is defined as follows, using (6) and (7):

$$\lambda_u = 2 - \lim_{t\to 0^+}(\frac{1-\varphi^{-1}(2t)}{1-\varphi^{-1}(t)}) = 2 - 2^{\frac{1}{\theta}}, \tag{8}$$

$$\lambda_l = \lim_{t\to\infty}(\frac{\varphi^{-1}(2t)}{\varphi^{-1}(t)}) = 2^{-\frac{1}{\theta}}. \tag{9}$$

The Gumble family is known to have only upper tail dependence while the Clayton family has only lower tail dependence. Since the tail dependence coincide, the Cot family has the same upper tail dependence as Gumble with exactly the same bound and the same lower tail dependence as Clayton copula in range of [1/2,1].

Kendall's τ

For Archimedean copulas, the Kendall's τ can be expressed in terms of the generator φ as:

$$\tau_{X_1,X_2} = \tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt = 1 - 4\int_0^\infty u \left[\frac{d}{du}\varphi^{-1}(u)\right]^2 du. \quad (10)$$

The Kendall's τ for the generator $\varphi(t) = \cot^\theta(\frac{\pi}{2}x)$ is then given by:

$$\tau_{X_1,X_2} = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt = 1 + 4[\int_0^1 \frac{\cot^\theta(\frac{\pi}{2}t)}{(-\theta\frac{\pi}{2})\cot^{\theta-1}(\frac{\pi}{2}t)(1+\cot^2(\frac{\pi}{2}t))} dt] = 1 - \frac{8}{\pi^2}(\frac{1}{\theta}). \quad (11)$$

Thus, the Cot-copula function has a range of dependency between $[1-\frac{8}{\pi^2}(\frac{1}{\theta}),1]$.

According to Table 4.1 of (Nelsen, 2006), the 12th families of Archimedean copula also have both upper and lower tail dependence. The dependence coverage for 12th family is:.

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt = 1 + 4[\int_0^1 \frac{(\frac{1}{t}-1)^\theta}{(-\theta)(\frac{1}{t}-1)^{(\theta-1)}(\frac{1}{t})^2} dt = 1 - \frac{4}{6}(\frac{1}{\theta}). \quad (12).$$

According to formula (12) the dependence coverage for 12th family is [0.34, 1] when $\lim_{\theta->1}(1-\frac{4}{6\theta}) = 0.34$ and $\lim_{\theta->\infty}(1-\frac{4}{6\theta}) = 1$. However, the Cot- family has dependence coverage of [0.19, 1] from (11). Thus, in comparison, the Cot-copula has wider dependence coverage rather than 12th family which illustrates the goodness of the proposed copula in capturing a wider dependence structure.

Empirical application

Measuring tail dependence on an asymmetric data using the cot- copula is the objective of this section. 1000 observations are generated from an asymmetric distribution function with both tail dependences. Firstly, marginal distribution functions are independently estimated via nonparametric kernel estimation method. After transforming the standardized residuals into uniform margins, three copula functions, Gumbel, Clayton and Cot-copula have been fitted. Estimated parameters with standard errors based on Kendall's process, Archimedean goodness of fit method, is listed on Table 2. Cot-Copula seems to show a good performance on both tail dependences at one time with only one parameter.

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Table 1: tail dependence compartion

	λ_l	λ_u
Gumble (θ≥1)	0	$2-2^{\frac{1}{\theta}}$
Clayton (θ≥0)	$2^{-\frac{1}{\theta}}$	0
Cot (θ≥1)	$2^{-\frac{1}{\theta}}$	$2-2^{\frac{1}{\theta}}$

Table 2: tail dependence estimated for two sets of data generate from an asymmetric distribution function.

C^C		C^G		C^{Cot}		
θ	λ_L	θ	λ_U	θ	λ_U	λ_L
0.1488 [0.023]	0.0266	1.0898 [0.013]	0.1111	1.4282 [0.077]	0.1326	0.0176
0.7081 [0.032]	0.3757	1.3654 [0.019]	0.3386	1.7946 [0.091]	0.3496	0.3519

[Standard errors are given in square brackets].