Exponential stopping of jump-diffusion processes and applications

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1 Exponential stopping of a jump-diffusion process

Let

$$X(t) = \mu t + \sigma W_t - \sum_{i=1}^{N_1(t)} Y_i + \sum_{i=1}^{N_2(t)} V_i, \tag{1.1}$$

where W_t is a standard Brownian motion, $\mu > 0$ is the drift and $\sigma > 0$ is the diffusion parameter, $N_1(t)$ is a Poisson process with parameter ω and $(Y_i,\ i = 1, 2, ...)$ is a sequence i.i.d exponential random variables, the probability density function (pdf) of each is denoted as $q(x),\ x > 0$. $N_2(t)$ is an independent Poisson process with parameter ν, and $(V_i,\ i = 1, 2, ...)$ is a sequence i.i.d exponential random variables, the pdf of each is denoted as $p(x),\ x > 0$. Thus

$$\mathrm{E}[e^{zX(t)}] = e^{t\Psi(z)}, \qquad -w < z < v, \tag{1.2}$$

with

$$\Psi(z) \quad = \quad Dz^2 + \mu z + \nu \frac{z}{v - z} - \omega \frac{z}{w + z}. \tag{1.3}$$

Here

$$D = \frac{1}{2}\sigma^2. \tag{1.4}$$

Because

$$\mathrm{E}[e^{zX(\tau)}] \quad = \quad E[E[e^{zX(\tau)}|\tau]] = \mathrm{E}[e^{\tau\Psi(z)}] = \frac{\lambda}{\lambda - \Psi(z)}, \tag{1.5}$$

the mgf of $X(\tau)$ is

$$\frac{\lambda}{-Dz^2 - \mu z + \lambda - \nu\frac{z}{v-z} + \omega\frac{z}{w+z}}, \tag{1.6}$$

*Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, e-mail: hlyang@hku.hk. This is a joint paper with Hans U. Gerber and Elias S.W. Shiu a rational function of z. For identifying the distribution of $X(\tau)$, we apply the method of partial fractions and write (1.6) as

$$\begin{aligned}&\frac{\lambda}{D}\frac{(v-z)(w+z)}{(z-\alpha_1)(z-\alpha_2)(z-\beta_1)(z-\beta_2)}\\&=\frac{\lambda}{D}\left(\frac{A_1}{z-\alpha_1}+\frac{A_2}{z-\alpha_2}-\frac{B_1}{z-\beta_1}-\frac{B_2}{z-\beta_2}\right).\end{aligned} \tag{1.7}$$

Here α_1, α_2, β_1 and β_2 are the four zeros of the denominator in (1.6). They are separated by $-w$, 0, v. We make the convention

$$-\infty<\alpha_2<-w<\alpha_1<0<\beta_1<v<\beta_2<\infty. \tag{1.8}$$

Then the probability density function of $X(\tau)$ is

$$f_{X(\tau)}(x)=\begin{cases}\frac{\lambda}{D}(A_1e^{-\alpha_1 x}+A_2e^{-\alpha_2 x}), & \text{if } x\le 0,\\ \frac{\lambda}{D}(B_1e^{-\beta_1 x}+B_2e^{-\beta_2 x}), & \text{if } x>0.\end{cases} \tag{1.9}$$

The coefficients are found to be as follows:

$$\begin{aligned}A_1&=\frac{(v-\alpha_1)(w+\alpha_1)}{(\alpha_1-\alpha_2)(\beta_1-\alpha_1)(\beta_2-\alpha_1)},\\A_2&=\frac{-(v-\alpha_2)(w+\alpha_2)}{(\alpha_1-\alpha_2)(\beta_1-\alpha_2)(\beta_2-\alpha_2)},\\B_1&=\frac{(v-\beta_1)(w+\beta_1)}{(\beta_1-\alpha_1)(\beta_1-\alpha_2)(\beta_2-\beta_1)},\\B_2&=\frac{(\beta_2-v)(w+\beta_2)}{(\beta_2-\alpha_1)(\beta_2-\alpha_2)(\beta_2-\beta_1)}.\end{aligned} \tag{1.10}$$

To verify, for example, the expression given for A_1, it suffices to multiply (1.7) by $z-\alpha_1$, simplify, and set $z=\alpha_1$. Furthermore, it can be verified that $A_1+A_2=B_1+B_2$. Thus $f_{X(\tau)}(x)$ is continuous at $x=0$.

2 Distribution of $M(\tau)$ when $p(x)$ is exponential

In this section, we assume that the jump diffusion $\{X(t);\ t\ge 0\}$ has upward jumps with exponential jump size. No specific assumption about $q(x)$ (the pdf of the downward jumps) is needed. Thus

$$\Psi(z)=Dz^2+\mu z+\nu\frac{z}{v-z}-\omega\int_0^\infty(1-e^{-zx})q(x)dx. \tag{4.1}$$

Then equation $\Psi(z)=\lambda$ has exactly two positive roots β_1 and β_2, with $0<\beta_1<v<\beta_2<\infty$, and

$$\Pr(M(\tau)\ge x)=\frac{\beta_2(v-\beta_1)e^{-\beta_1 x}+\beta_1(\beta_2-v)e^{-\beta_2 x}}{v(\beta_2-\beta_1)},\quad x\ge 0, \tag{4.2}$$

where $M(\tau)$ is the maximum of the jump diffusion $\{X(t);\ t\ge 0\}$ up to time τ. Thus

$$f_{M(\tau)}(x)=\frac{\beta_2(v-\beta_1)}{v(\beta_2-\beta_1)}\beta_1e^{-\beta_1 x}+\frac{\beta_1(\beta_2-v)}{v(\beta_2-\beta_1)}\beta_2e^{-\beta_2 x},\qquad x\ge 0, \tag{4.3}$$

a mixture of two exponential pdf's.

3 Distribution of $m(\tau)$ when $q(x)$ is exponential

In this section, we assume that the jump diffusion $\{X(t);\ t \geq 0\}$ has downward jumps with exponential jump size. No specific assumption is made for $p(x)$ (the pdf of the upward jumps). Thus

$$\Psi(z) = Dz^2 + \mu z + \nu \int_0^\infty (e^{zx} - 1)p(x)dx - \omega \frac{z}{w+z}. \tag{5.1}$$

Equation $\Psi(z) = \lambda$ has now exactly two negative roots α_1 and α_2 with $-\infty < \alpha_2 < -w < \alpha_1 < 0$. The running minimum of the process $\{X(t);\ t \geq 0\}$ is the same as the negative of the running maximum of the process $\{-X(t);\ t \geq 0\}$. From this and (4.2) it follows that

$$\Pr(m(\tau) \leq x) = \frac{-\alpha_2(w+\alpha_1)e^{-\alpha_1 x} + \alpha_1(w+\alpha_2)e^{-\alpha_2 x}}{w(\alpha_1 - \alpha_2)}, \quad x \leq 0, \tag{5.2}$$

and hence the pdf of $m(\tau)$ is the following mixture of the exponential pdf's:

$$f_{m(\tau)}(x) = \frac{-\alpha_2(w+\alpha_1)}{w(\alpha_1-\alpha_2)}(-\alpha_1 e^{-\alpha_1 x}) + \frac{\alpha_1(w+\alpha_2)}{w(\alpha_1-\alpha_2)}(-\alpha_2 e^{-\alpha_2 x}), \quad x \leq 0. \tag{5.3}$$

For each $t > 0$, the random variables $M(t) - X(t)$ and $|m(t)| = -m(t)$ have the same distribution due to the property of stationary increments. It follows that

$$\begin{aligned} & f_{M(\tau)-X(\tau)}(x) \\ &= f_{m(\tau)}(-x) \\ &= \frac{-\alpha_2(w+\alpha_1)}{w(\alpha_1-\alpha_2)}(-\alpha_1 e^{\alpha_1 x}) + \frac{\alpha_1(w+\alpha_2)}{w(\alpha_1-\alpha_2)}(-\alpha_2 e^{\alpha_2 x}), \quad x \geq 0. \end{aligned} \tag{5.4}$$

4 Bivariate distributions

In this section we assume again the double exponential jump diffusion model that was introduced in Section 1. Because $\{X(t);\ t \geq 0\}$ is a Lévy process and τ is memoryless, the random variables $M(\tau)$ and $M(\tau) - X(\tau)$ are independent. Thus their joint pdf can be factorized,

$$f_{M(\tau),M(\tau)-X(\tau)}(y,z) = f_{M(\tau)}(y) f_{M(\tau)-X(\tau)}(z), \qquad y \geq 0,\ z \geq 0, \tag{6.1}$$

with $f_{M(\tau)}$ and $f_{M(\tau)-X(\tau)}$ given by (4.3) and (5.4).

Because

$$X(\tau) = M(\tau) - [M(\tau) - X(\tau)], \tag{6.2}$$

the joint pdf of $X(\tau)$ and $M(\tau)$, for $y \geq \max(0, x)$, is

$$\begin{aligned} f_{X(\tau),M(\tau)}(x,y) &= f_{M(\tau),M(\tau)-X(\tau)}(y, y-x) \\ &= f_{M(\tau)}(y)\ f_{M(\tau)-X(\tau)}(y-x) \\ &= \frac{\lambda}{D(\alpha_1-\alpha_2)(\beta_2-\beta_1)}\ \{-\}, \end{aligned} \tag{6.3}$$

with

$$\begin{aligned} \{-\} &= (w+\alpha_1)(v-\beta_1)e^{-\alpha_1 x}e^{(\alpha_1-\beta_1)y} - (w+\alpha_2)(v-\beta_1)e^{-\alpha_2 x}e^{(\alpha_2-\beta_1)y} \\ &+ (w+\alpha_1)(\beta_2-v)e^{-\alpha_1 x}e^{(\alpha_1-\beta_2)y} - (w+\alpha_2)(\beta_2-v)e^{-\alpha_2 x}e^{(\alpha_2-\beta_2)y}. \end{aligned} \quad (6.4)$$

Here $f_{M(\tau)}$ and $f_{M(\tau)-X(\tau)}$ have been substituted from (4.3) and (5.4), and we used that

$$\alpha_1\alpha_2\beta_1\beta_2 = \frac{\lambda v w}{D}. \quad (6.5)$$

To obtain the last relation, we set $z = 0$ in the LHS of (1.7) and remember that a mgf has the value 1 for $z = 0$.

The formula for $f_{X(\tau),m(\tau)}(x, y)$, for $y \le \min(x, 0)$, is

$$f_{X(\tau),m(\tau)}(x,y) = \frac{\lambda}{D(\alpha_1-\alpha_2)(\beta_2-\beta_1)}\,\{-\}, \quad (6.6)$$

with

$$\begin{aligned} \{-\} &= (w+\alpha_1)(v-\beta_1)e^{-\beta_1 x}e^{(\beta_1-\alpha_1)y} - (w+\alpha_2)(v-\beta_1)e^{-\beta_1 x}e^{(\beta_1-\alpha_2)y} \\ &+ (w+\alpha_1)(\beta_2-v)e^{-\beta_2 x}e^{(\beta_2-\alpha_1)y} - (w+\alpha_2)(\beta_2-v)e^{-\beta_2 x}e^{(\beta_2-\alpha_2)y}. \end{aligned} \quad (6.7)$$

Remark 6.1: Formulas (6.1) and (6.2) can be used for an alternative derivation of the pdf of $X(\tau)$. We have

$$f_{X(\tau)}(x) = \begin{cases} \int_0^\infty f_{M(\tau),M(\tau)-X(\tau)}(y, y-x)dy, & x < 0, \\ \int_0^\infty f_{M(\tau),M(\tau)-X(\tau)}(z+x, z)dz, & x \ge 0. \end{cases} \quad (6.8)$$

We substitute from (6.1), (4.3) and (5.4). After some calculation, we find that

$$f_{X(\tau)}(x) = \begin{cases} a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x}, & x < 0, \\ b_1 e^{-\beta_1 x} + b_2 e^{-\beta_2 x}, & x \ge 0 \end{cases} \quad (6.9)$$

with

$$\begin{aligned} a_1 &= \frac{\alpha_1\alpha_2\beta_1\beta_2(v-\alpha_1)(w+\alpha_1)}{vw(\alpha_1-\alpha_2)(\beta_1-\alpha_1)(\beta_2-\alpha_1)}, \\ a_2 &= \frac{-\alpha_1\alpha_2\beta_1\beta_2(v-\alpha_2)(w+\alpha_2)}{vw(\alpha_1-\alpha_2)(\beta_1-\alpha_2)(\beta_2-\alpha_2)}, \\ b_1 &= \frac{\alpha_1\alpha_2\beta_1\beta_2(v-\beta_1)(w+\beta_1)}{vw(\beta_1-\alpha_1)(\beta_1-\alpha_2)(\beta_2-\beta_1)}, \\ b_2 &= \frac{\alpha_1\alpha_2\beta_1\beta_2(\beta_2-v)(w+\beta_2)}{vw(\beta_2-\alpha_1)(\beta_2-\alpha_2)(\beta_2-\beta_1)}. \end{aligned} \quad (6.10)$$

To reconcile this result with (1.9) and (1.10), apply (6.5).

5 Contingent options

In the following, the price of one unit of a stock (or of a stock fund) at time t is denoted as $S(t)$. It is assumed that

$$S(t) = S(0)e^{X(t)}, \qquad t \geq 0, \tag{7.1}$$

where $\{X(t);\ t \geq 0\}$ is the double exponential jump diffusion discussed in Section 3. Thus we know the pdf of the random variable $X(\tau)$. We consider a contract that is defined by a benefit function $b(s)$: it provides a payment of $b(S(\tau))$ at time τ. The benefit function for a K-strike put option is $b(s) = (K - s)_+$, and the one for a K-strike call is $b(s) = (s - K)_+$.

We are interested in the expected discounted value of the benefit payment,

$$\begin{aligned} V(r, \lambda) &= \mathrm{E}[e^{-r\tau} b(S(\tau))] \\ &= \lambda \int_0^\infty e^{-(\lambda+r)t} \mathrm{E}[b(S(t))] dt. \end{aligned} \tag{7.2}$$

Here r is an appropriate continuously compounded rate of return. It follows from (7.2) that

$$V(r, \lambda) = \frac{\lambda}{\lambda + h} V(r - h, \lambda + h) \tag{7.3}$$

for any $h > -\lambda$. In particular, for $h = r$,

$$V(r, \lambda) = \frac{\lambda}{\lambda + r} V(0, \lambda + r). \tag{7.4}$$

This identity shows that it suffices to have formulas and results for $V(0, \lambda)$, the time-0 expectation of the benefit payment. Then, the time-0 expected discounted value of the payment, $V(r, \lambda)$, is obtained from the substitution $\lambda \leftarrow \lambda + r$ and multiplication by $\frac{\lambda}{\lambda+r}$.

From now on, we assume the interest rate is 0. Setting $r = 0$ in (7.2), we have

$$V(0, \lambda) = \int_{-\infty}^{\infty} b(S(0)e^x) f_{X(\tau)}(x) dx, \tag{7.5}$$

where $f_{X(\tau)}(x)$ is given by (1.9) or (6.9). When $b(S(\tau)) = (S(\tau) - K)_+$, $V(0, \lambda)$ is the call option value.

More generally,We are interested in the expected discounted value of the benefit payment,

$$\begin{aligned} V(r, \lambda) &= \mathrm{E}[e^{-r\tau} b(S(\tau), M(\tau))] \\ &= \frac{\lambda}{\lambda + r} V(0, \lambda + r). \end{aligned} \tag{7.6}$$

We can specify the functional $b(s, m)$ to obtain values of lookback, and barrier options.